HU, Probability Theory Fall 2015, Problems 4 (30.9.2015)

1. Prove that if $\mathbb{P}$ is a probability measure on $\Omega = \mathbb{R}^d$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, every Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies the following **Approximation Property** : for every $\varepsilon > 0$ there is an open set $U \subseteq \mathbb{R}^d$ and a closed set $C \subseteq \mathbb{R}^d$ such that $U \supseteq B \supseteq C$ and $\mathbb{P}(U \setminus C) \leq \varepsilon$.

   To show that consider the class of events

   $$\mathcal{D} = \big\{ B \in \mathcal{B}(\mathbb{R}^d) \text{ which has the Approximation property } \big\} \subseteq \mathcal{B}(\mathbb{R}^d).$$

   - Show first that the class

     $$\mathcal{C} = \{\ C \subseteq \mathbb{R}^d,\ C \text{ closed } \} \subset \mathcal{D}$$

     and it is a π-class (closed under intersections).

     Hint if C is closed, let $C^\varepsilon = \{\ y : \exists x \in C \text{ with } |x - y| < \varepsilon\} \supseteq C$ Show that C^ε is open and

     $$C = \bigcap_{n \in \mathbb{N}} C^{1/n}$$

     Use the σ-addivity of $\mathbb{P}$ to show that C has the Approximation propery.
   - Then show that $\mathcal{D}$ is a Dynkin class.
   - Use Dynkin lemma to conclude that all Borel sets have the Approximation property.
   - Prove also that when $B \in \mathcal{B}(\mathbb{R}^d)$ is a Borel set, $\forall \varepsilon > 0$ one can find an open set U and a compact set K with $U \geq B \geq K$ and $\mathbb{P}(U \setminus K) < \varepsilon$.

   Remark We have used the Approximation property of the Borel sets in the proof of Kolmogorov extension Theorem.

2. Let $f : \mathbb{R} \to \mathbb{R}$ a non-decreasing function, $f(s) \leq f(t)$ when $s \leq t$. Show that f is Borel measurable, which means that for every Borel set $B \in \mathcal{B}(\mathbb{R})$, the counterimage $f^{-1}(B) := \{\ t : f(t) \in B\}$ is a Borel set.

3. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_n : n \in \mathbb{N})$ be any sequence of pairwise disjoint events, which means $A_i \cap A_j = \emptyset$ when $i \neq j$. Show that $\lim\limits_{n \to \infty} \mathbb{P}(An) = 0$.

4. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(A_\alpha, \alpha \in I)$ be a family of pairwise disjoint events, indexed by an index set I. Show that if $\mathbb{P}(A_\alpha) > 0$ $\forall \alpha \in I$, then I must be countable.

   Hint: show that $\forall n \in \mathbb{N}$ the set $\{\alpha : \mathbb{P}(A_\alpha > 1/n\}$ is finite .

5. Let P a probability on $\Omega = \mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. We have shown the *cumulative distribution function* $F(t) = P\big((-\infty, t]\big)$ is right-continuous, which means

$$F(t+) = \lim_{u \downarrow t} F(u) = F(t) \quad \forall t \in \mathbb{R} .$$

   Denote the jump size of F at t by $\Delta F(t) = F(t) - F(t-)$ where $F(t-) = \lim_{s \uparrow t} F(s)$ is the limit from the left.

   (a) Show that $P(\{t\}) = \Delta F(t)$.

   (b) Show that the set $\Big\{ t \in \mathbb{R} : \ \Delta F(t) > 0 \Big\}$ is at most a countable set.

6. Suppose a function $F : \mathbb{R} \to [0, 1]$ is given by

$$F(t) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}(t \geq 1/n)$$

   (a) Show that $F(t)$ the cumulative distribution function of a probability P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

   For such P, compute the probabilities of the following events:

   - $A = [1, \infty)$,
   - $B = [1/10, \infty)$,
   - $C = \{ \ 0\}$,
   - $D = [0, 1/2)$,
   - $E = (-\infty, 0)$,
   - $G = (0, \infty)$.

   (b) Define a random variable X on a probabilty space $(\Omega, \mathcal{F}, \mathbb{P})$ of your choice, with a probability $\mathbb{P}$ of your choice, such that the distribution $\mathbb{P}(\{\omega : X(\omega) \leq t\}) = F(t)$.

   Hint: you can always define the random variable as the identity map in the space where it takes values, in this case $\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.