HU, Probability Theory Fall 2015, Problems 3 (23.9.2015)
(On the Cylinder algebra on an infinite product space).

Let S be an abstract probability space equipped with a σ-algebra $\mathcal{S}$, for example $S = \mathbb{R}^d$ and $\mathcal{S} = \mathcal{B}(\mathbb{R}^d)$, the Borel σ-algebra. and T an (infinite) arbitrary set . Consider the space $\Omega = S^T$, whose elements are the maps $\omega : T \to S$, with $t \mapsto \omega_t \in S$.

We can also understand Ω as the infinite product space $\Omega = \prod_{t\in T} S_t$, where each S_t is a copy of S.

A cylinder is an Ω-subset with representation

$$C = \big\{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{t_d}) \in B_{t_1 \dots t_d}\big\} \tag{0.1}$$

for some $d \in \mathbb{N}$, $t_1, \dots, t_d \in T$ and $B_{t_1 \dots t_d} \in \mathcal{S}^{\otimes d} = \underbrace{\mathcal{S} \otimes \mathcal{S} \otimes \dots \otimes \mathcal{S}}_{d\text{-times}}$, the d-fold product of σ-algebrae. In other words, whether a function ω belongs to a cylinder C or not it is determined by its values on a finite number of coordinates.

Note that the cylinder representation (0.1) is not unique, for example the same cylinder C could be expressed as

$$C = \big\{\omega : (\omega_{t_1}, \omega_{t_2} \dots, \omega_{t_d}, \omega_{t_{d+1}}) \in B_{t_1 \dots t_d} \times S\big\}$$

Q$_1$: Show that the cylinders $\mathcal{C} = \{C \subseteq \Omega : C \text{ is a cylinder } \}$ form an algebra of Ω-events.

Q$_2$: However, the cylinders do not form a σ-algebra when T is infinite. Find an example where the countable intersection of cylinders is not a cylinder.

A consistent family $\mathcal{P}$ of finite dimensional distribution is a collection of probability measures $P_{t_1,\dots,t_d}$ on the respective product σ-algebrae $\mathcal{S}^{\otimes d}$ indexed by $t_1, t_2, \dots, t_d \in T$, where d varies in $\mathbb{N}$, satisfying the properties:

- $$P_{t_1,\dots,t_d}\big(B_{t_1} \times \dots \times B_{t_d}\big) = P_{t_{\pi(1)},\dots,t_{\pi(d)}}\big(B_{t_{\pi(1)}} \times \dots \times B_{t_{\pi(d)}}\big) =$$

  for every d, $t_1, \dots, t_d \in T$ and π permutation of $\{\ 1, 2, \dots, d\}$, and $B_{t_i} \in \mathcal{S}$.

- $$P_{t_1,\dots,t_d}\big(B_{t_1,\dots,t_d}\big) = P_{t_1,\dots,t_d,t_{d+1}}\big(B_{t_1,\dots,t_d} \times S\big) =$$

  $\forall d,\ t_1,\dots,t_d,t_{d+1} \in T$ and $B_{t_1\dots,t_d} \in \mathcal{S}^{\otimes d}$.

Q$_3$: Show that the map

$$\mathbb{P}_0 : \mathcal{C} \to [0,1]$$

with $\mathbb{P}_0(C) = P_{t_1\dots t_d}(B_{t_1\dots t_d})$ for C with representation (0.1) is well defined, meaning that it does not depend on the particular representation of the cylinder C, and that $\mathbb{P}^0$ is finitely additive on the algebra $\mathcal{C}$.

For each t, let Q_t a probability on $(S,\mathcal{S})$.

Define the family $\mathcal{Q}$ of finite dimensional distributions

$Q_{t_1\dots t_d} = Q_{t_1} \otimes Q_{t_2} \otimes \dots \otimes Q_{t_d}$ as the product measure on the product space S^d equipped with product σ-algebra $\mathcal{S}^{\otimes d}$.

Q$_4$: Show that $\mathcal{Q}$ is a consistent family of finite dimensional distributions.

Remark The next question which will be adressed in the lectures is: can we extend uniquely $\mathbb{P}^0$ to a σ-additive probability defined on the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinders ? By Caratheordory theorem, it is enough to show that $\mathbb{P}^0$ is σ-additive on the cylinder algebra, namely if $(C_n : n \in \mathbb{N}) \subset \mathcal{C}$ is a cylinder sequence with $C_n \downarrow \emptyset$, necessarily $\mathbb{P}^0(C_n) \downarrow 0$. This is the content of Kolmogorov extension theorem, which requires an additional assumption on the probability space $(S,\mathcal{S})$.

Q$_5$: In general, let Ω an abstract space and $\mathcal{E} \subseteq 2^\Omega$ a collection of Ω-subsets. Let $\mathcal{F} = \sigma(\mathcal{E})$ the σ-algebra generated by $\mathcal{E}$.

Show that $A \in \mathcal{F}$ if and only if $A \in \sigma(\mathcal{C})$ for some countable collection $\mathcal{C} \subseteq \mathcal{E}$, which may depend on A.

Hint: Show that the set

$$\big\{A \in \mathcal{F} : A \in \sigma(\mathcal{C}) \text{ for some countable } \mathcal{C} \subseteq \mathcal{E}\ \big\}$$

is both a π-class and a Dynkin class and it contains $\mathcal{E}$.

Q$_6$: We come back to the construction of the σ-algebra generated by the cylinders on $\Omega = S^T$. Using the previous exercise, show that a set A in the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinders is determined by at most countably many T-coordinates.

In particular, when $T = \mathbb{R}^m$ and $S = \mathbb{R}^d$, show that the space of continuous function

$$C(\mathbb{R}^m, \mathbb{R}^d) = \big\{\omega : \mathbb{R}^m \to \mathbb{R}^d \text{ continuous functions } \big\} \subseteq (\mathbb{R}^d)^{\mathbb{R}^m}$$

is not in the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinders.