HU, Probability Theory Fall 2015, Problems 2 (16.9.2015)

1. A finitely additive probability $\mathbb{P}$ on a probability space Ω equipped with a σ-algebra $\mathcal{F}$ is also σ-additive if and only if for any event sequence $(A_n : n \in \mathbb{N}) \subseteq \mathcal{F}$ with $A_n \downarrow \emptyset$, meaning that $\forall n \in \mathbb{N}\ A_n \supseteq A_{n+1}$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, it follows that $\mathbb{P}(A_n) \downarrow 0$.

   This does not hold for infinite measures with $\lambda(\Omega) = \infty$, for example for the Lebesgue measure λ on $\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, such that $\lambda((a, b]) = (b - a)^+$. Here $x^+ = \max\{x, 0\}$ is the notation for the positive part of $x \in \mathbb{R}$.

   Find a counterexample, as a sequence $(A_n : n \in \mathbb{N}) \subset \mathcal{B}(\mathbb{R})$, with $A_n \downarrow 0$ but $\lambda(A_n) \not\to 0$.

2. Let $\Omega = \mathbb{R}^d$, the euclidean space. In general the Borel σ algebra is the smallest σ-algebra containing the open sets.

   For $t \in \mathbb{R}^d$, we introduce the infinite rectangle

   $$(-\infty, t] = \{s \in \mathbb{R}^d :\ s_i \le t_i, i = 1, \dots d\}$$

   Show that the class

   $$\mathcal{I} = \left\{(-\infty, q], q \in \mathbb{Q}^d\right\}$$

   is a π-class with $\sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R}^d)$.

   Hint If U is open in $\mathbb{R}^d$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall x \in U\ \exists r, q \in \mathbb{Q}^d$ such that $r < q$ (meaning that $r_i < q_i$ for each coordinate $i = 1, \dots, d$

   $$x \in (r, q) := (r_1, q_1) \times (r_2, q_2) \times \cdots \times (r_d, q_d) \subseteq U.$$

   i.e. there is a small open rectangle containg x which is contained in U.

3. Consider a probabilty triple $(\Omega, \mathcal{F}, \mathbb{P})$, and a sequence of events $A_n \in \mathcal{F}$ such that $\mathbb{P}(A_n) = 1\ \forall n \in \mathbb{N}$.

   Show that $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = 1$.

   Consider also a sequence $B_n \in \mathcal{F}$ such that $\mathbb{P}(B_n) = 0\ \forall n \in \mathbb{N}$.

   Show that $\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = 0$.

4. Consider the probability space $(\Omega, \mathcal{F})$ and an event sequence $(A_n : n \in \mathbb{N}) \subseteq \mathcal{F}$. We denote

$$\limsup_n A_n := \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_k, \quad \liminf_n A_n := \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} A_k,$$

   (a) Show that $(\limsup_n A_n) \in \mathcal{F}$.
   (b) Show that $(\liminf_n A_n) = (\limsup_n A_n^c)^c$. where $B^c = \Omega \setminus B$ is the complement event.
   (c) Show that $(\liminf_n A_n) \in \mathcal{F}$.
   (d) For $A \in \mathcal{A}$, let $\mathbf{1}_A(\omega)$, the indicator function of A defined for $\omega \in \Omega$. Show that

$$\mathbf{1}_{(\limsup_n A_n)}(\omega) = \limsup_n \mathbf{1}_{A_n}(\omega), \quad \mathbf{1}_{(\liminf_n A_n)}(\omega) = \liminf_n \mathbf{1}_{A_n}(\omega)$$

   (e) Show that

$$\limsup_n A_n = \{\omega : \omega \in A_n \text{ infinitely often, which means for infinitely many } n \ \}$$

$$\liminf_n A_n = \{\omega : \omega \in A_n \text{ eventually, which means for all } n \text{ large enough } \}$$

   (f) Let $A_n \subseteq B_n \ \forall n$. Show that

$$\limsup_n A_n \subseteq \limsup_n B_n, \quad \liminf_n A_n \subseteq \liminf B_n$$

   (g) Suppose that for a probability $\mathbb{P}$ we have $\mathbb{P}(\limsup_n A_n) = 1$ and $\mathbb{P}(\liminf_n B_n) = 1$. Show that $\mathbb{P}(\limsup_n A_n \cap B_n) = 1$.

5. Let $(\Omega, \mathcal{F})$ a probability space, with a sequence of probability measures $(\mathbb{P}_n : n \in \mathbb{N})$.

   Suppose that $\forall A \in \mathcal{F}$ the limits

$$\mathbb{P}(A) := \lim_{n\to\infty} \mathbb{P}_n(A)$$

   exists.

   (a) Prove that in such case the map $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ is a probability measure.
   (b) For each event sequence $(A_k : k \in \mathbb{N}) \subseteq \mathcal{F}$ such that $A_k \downarrow \emptyset$ we have

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n(A_k) \downarrow 0 \text{ as } k \uparrow \infty$$