HU, Probability Theory Fall 2015, Problems 1 (9.9.2015)

1. Let $\Omega = [0,1] \cap \mathbb{Q} = \{r \text{ rational} : 0 \le r \le 1 \}$,

   and $\mathcal{A}$ the collection of sets which can be represented as finite unions of intervals of type $(a,b] \cap \mathbb{Q}$, $[a,b] \cap \mathbb{Q}$,$(a,b) \cap \mathbb{Q}$, or $[a,b) \cap \mathbb{Q}$, with $0 \le a \le b \le 1$.

   Define $\forall 0 \le a \le b \le 1$

   $$P((a,b] \cap \mathbb{Q}) = P([a,b] \cap \mathbb{Q}) = P((a,b) \cap \mathbb{Q}) = P([a,b) \cap \mathbb{Q}) = b - a,$$

   - Show $\mathcal{A}$ is an algebra, which means $\Omega \in \mathcal{A}$, and when $A \in \mathcal{A}$ also $A^c := (\Omega \setminus A) \in \mathcal{A}$ and if $A, B \in \mathcal{A}$ also $A \cup B \in \mathcal{A}$.
   - Extend the function P to a finitely additive probability on the algebra $\mathcal{A}$.
   - Show that such additive P is not σ-additive.

   Hint $\Omega = [0,1] \cap \mathbb{Q}$ is countable !.

2. Consider an abstract set Ω, and define the collection

   $$\mathcal{A} = \Big\{ A \subseteq \Omega : \text{ either } A \text{ or its complement } A^c = \Omega \setminus A \text{ is finite } \Big\}$$

   - Show that $\mathcal{A}$ is an algebra but it is not a σ-algebra.
   - For $A \in \mathcal{A}$, define $Q(A) = 0$ when A is finite and $Q(A) = 1$ when A is infinite. Show that Q is finitely additive on $\mathcal{A}$ but not σ-additive.

3. Let Ω an abstract set and 2^Ω its power, which is the collection of subsets $A \subseteq \Omega$.

   Define the symmetric difference of $A, B \subseteq \Omega$ as

   $$A \Delta B = (A \cup B) \setminus (A \cap B) = \{\omega : \omega \in A \text{ or } \omega \in B \text{ but not in both } \}$$

   Show that 2^Ω is a **ring** with respect to the operations Δ (sum) and $\cap$ (product), which means

   - Find an identity element with respect to the operation Δ.
   - Find an identity element with respect to the operation $\cap$.
   - Show that every element $A \subset \Omega$ has an additive inverse,

- Show that Δ is associative and the distributive property holds between Δ and $\cap$.

Hint : for indicators we have

$$\mathbf{1}_{A\cap B} = \mathbf{1}_A \mathbf{1}_B, \quad \mathbf{1}_{A\cup B} = 1_A + 1_B - 1_A 1_B$$
$$\mathbf{1}_{(A\Delta B)} = \big(\mathbf{1}_A + \mathbf{1}_B\big) \bmod 2 = \mathbf{1}_A + \mathbf{1}_B - 2\times \mathbf{1}_A\mathbf{1}_B = 1_A\mathbf{1}_{B^c} + \mathbf{1}_B\mathbf{1}_{A^c}$$

4. Consider an arbitrary collection of σ-algebrae $\{\mathcal{G}_\alpha : \alpha \in \mathcal{I}\}$ on the same set Ω.

   Show that the intersection of σ-algebrae

   $$\mathcal{G} := \bigcap_{\alpha\in\mathcal{I}} \mathcal{G}_\alpha$$

   is a σ-algebra.

5. About countable and uncountable sets:

   (a) Show that in the blackboard represented as $[0,1]^2$ there is place for an uncountable amount of mutually non-intersecting zero symbols 'O', (circles), where the circles can be also inside each other but they should not touch each other.

   (b) Show that on the blackboard $= [0,1]^2$ or on an infinite blackboard like $= \mathbb{R}^2$ there is place for at most a countable numbers of mutually non-intersecting '8' symbols, or ∞-symbols if you like, where the symbols can contain each other but the boundaries of different curves cannot touch each other.