1. Green's formulae and all that

1.1. The fundamental thm. of calculus

Recall: if $f \in C^1[a,b]$, then

$$\int_a^b f' \, dx = f(b) - f(a)$$

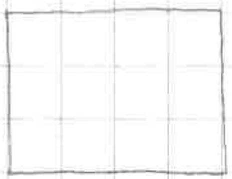
involves values of f only at end pts

Does this have an analogue in $\mathbb{R}^n$, $n > 1$?

Ans: Yes!

To see what it is, let's start with a trivial example:

$$\Omega = [a,b] \times [c,d]$$



We want to compute

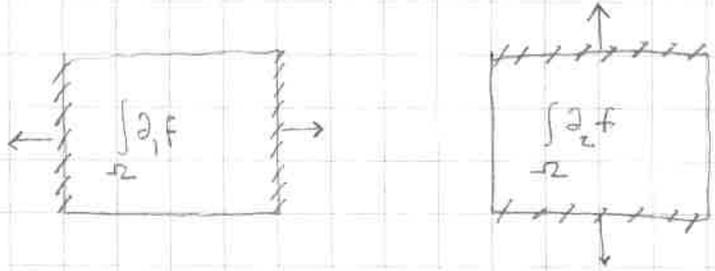
$$\int_\Omega \partial_1 f \, dx_1 dx_2 \quad \text{and} \quad \int_\Omega \partial_2 f \, dx_1 dx_2 .$$

$$\partial_i = \frac{\partial}{\partial x_i}$$

$$\int_\Omega \partial_1 f \, dx_1 dx_2 = \int_c^d \left(\int_a^b \frac{\partial f}{\partial x_1}(x_1,x_2) \, dx_1 \right) dx_2 = \int_c^d f(b,x_2) \, dx_2 - \int_c^d f(a,x_2) \, dx_2$$

Likewise

$$\int_\Omega \partial_2 f \, dx_1 dx_2 = \int_a^b f(x_1,d) \, dx_1 - \int_a^b f(x_1,c) \, dx_1$$

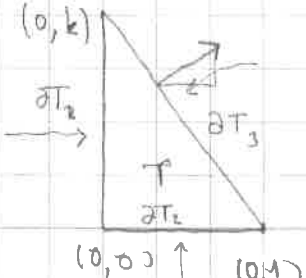


If $n = (n_1, n_2)$ the exterior unit normal to Ω (except at corners), we can write

$$\int_\Omega \partial_1 f \, dx_1 dx_2 = \int_{\partial\Omega} n_1 f \, dS , \qquad \int_\Omega \partial_2 f \, dx_1 dx_2 = \int_{\partial\Omega} n_2 f \, dS$$

has direction of $\pm(1,0)$ — has dir- of $\pm(0,1)$

This is not yet very convincing. Let's look at a triangle



$dS = dx_2$

$$x_2 = -kx_1 + k$$

$$dS = \sqrt{1 + \left(\frac{dx_2}{dx_1}\right)^2} \, dx_1 = \sqrt{1+k^2} \, dx_1$$

$$x_1 = -\frac{1}{k} x_2 + 1$$

$$dS = \sqrt{1 + \left(\frac{1}{k}\right)^2} \, dx_2 = \frac{1}{k}\sqrt{1+k^2} \, dx_2$$

$dS = dx_1$

$$x_2 = -kx_1 + k \qquad x_1 = \frac{x_2 - k}{-k} = -\frac{x_2}{k} + 1$$

Let's compute:

$$\iint_T \partial_1 f \, dx_1 dx_2 = \int_0^k \left(\int_0^{-\frac{x_2}{k}+1} \partial_1 f \, dx_1 \right) dx_2 = \int_0^k f\left(-\frac{x_2}{k}+1, x_2\right) dx_2$$

$$+ \int_0^k f(0, x_2)\, dx_2$$

$$= \int_{\partial T_3} f \frac{k}{\sqrt{1+k^2}} \, dS - \int_{\partial T_2} f \, dS$$

Now
$$n = \begin{cases} -(1,0) & \text{on } \partial T_2 \\ -(0,1) & \text{on } \partial T_1 \\ (k,1)/\sqrt{k^2+1} & \text{on } \partial T_3 \end{cases}$$

Hence
$$\iint_T \partial_1 f \, dx_1 dx_2 = \int_{\partial T} f n_1 \, dS.$$

Similarly
$$\iint_T \partial_2 f \, dx_1 dx_2 = \int_{\partial T} f n_2 \, dS$$

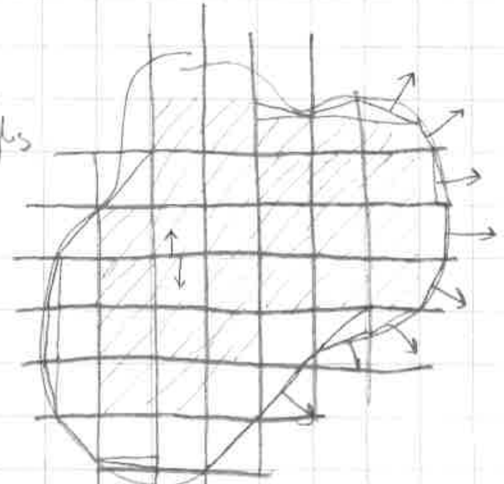
This holds for any triangle (use an affine change of coordinats)

$$\iint_T \partial_i f \, dx = \int_{\partial T} n_i f \, dS$$

Arb. domain

fill with small squares & triangles

integrals over common sides cancel (opposite normals)



Only integrals over free borders of triangles remain. Make this denser & denser: On the limit we get (& gen. to higher dim.)

Prop. If $\Omega \subset \mathbb{R}^d$ is a C^k-domain ($k \geq 1$), then $\forall f \in C^1(\Omega) \cap C(\bar{\Omega})$,

$$\int_\Omega \frac{\partial f}{\partial x_k} dx = \int_{\partial\Omega} n_k f \, dS \; ; \quad n = (n_1, \dots, n_d) \text{ ext. unit normal.}$$

1.2. Integration by parts.

We also have: Ω as above, $f, g \in C^1(\Omega) \cap C(\bar{\Omega})$,

$$\int_\Omega \frac{\partial f}{\partial x_k} g \, dx = \int_{\partial\Omega} n_k f g \, dS - \int_\Omega f \frac{\partial g}{\partial x_k} dS.$$

Pf. Fund. thm of calc. =>

$$\int_\Omega \frac{\partial f}{\partial x_k} g + f\frac{\partial g}{\partial x_k}\, dx \overset{\text{Prod. rule}}{=} \int_\Omega \frac{\partial (fg)}{\partial x_k} dx = \int_{\partial\Omega} n_k fg\, dS \qquad \square$$

1.3. Divergence thm.

Assume F is a C^1 vector field in Ω i.e.

$$F: \Omega \to \mathbb{R}^d, \quad F = (F_1, \dots, F_d), \quad F_k \in C^1(\Omega) \cap C(\bar\Omega).$$

Def. The divergence of F is

$$\operatorname{div} F(x) = \nabla\cdot F(x) = \sum_{k=1}^d \frac{\partial F_k(x)}{\partial x_k}.$$

Now we have

Thm. (Div. thm) If F is a C^1-vector field in $\Omega \subset \mathbb{R}^d$, Ω a C^1-domain, then

net flow of F through $\partial\Omega$.

$$\int_\Omega \nabla\cdot F\, dx = \int_{\partial\Omega} \langle n, F\rangle\, dS$$

$$\langle n, F\rangle = n_1F_1 + \dots + n_dF_d.$$

Pf. Fund. thm of calculus =>

$$\int_\Omega \frac{\partial F_k}{\partial x_k} dx = \int_{\partial\Omega} n_k F_k\, dS.$$

Sum this over k. $\square$

It is a good idea to think what the condition $\nabla F = 0$ in Ω means.

Let U be any (small) subdomain with C^1-bnd inside Ω. Then

$$\int_{\partial U} \langle n, F\rangle\, dS = \int_U \underbrace{\nabla\cdot F}_{=0}\, dx = 0.$$

∴ net flow of F through "any" surface inside Ω is 0!

i.e. no sinks or sources!

1.4. Gauss's thm.

Let $u \in C^2(\Omega) \cap C^1(\bar\Omega)$, Ω as above. Then gradient of u,

$$F = \nabla u$$

is a C^1-vector field in Ω.

Let's apply div. thm to this field. Now

$$\nabla \cdot F = \sum_k \frac{\partial F_k}{\partial x_k} = \sum_k \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_k}\right) = \sum_k \frac{\partial^2 u}{\partial x_k^2} = \Delta u \ !$$

i.e. div. thm =>

$$\int_\Omega \Delta u \, dx = \int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} \langle n, F \rangle dS = \int_{\partial\Omega} \langle n, \nabla u \rangle dS$$

$$= \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial n}}_{\text{normal deriv.}} dS$$

Hence we have proven Gauss-thm:

$$\boxed{\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS.}$$

This tells us something about the solvability of the Neumann-problem for Δ:

$$(N\text{-}\Delta) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g & \text{on } \partial\Omega. \end{cases}$$

Remember that $(N\text{-}\Delta)$ is <u>not</u> uniquely solvable:

If u is a sol., then so is $u + \text{const}$. ↑ Ω conn. ↓ "1-D cond"

Also, if g is such that $(N\text{-}\Delta)$ has a solution, then Gauss =>

$$\int_{\partial\Omega} g \, dS = \int_\Omega \overset{0}{\overset{\shortparallel}{\Delta u}} \, dx = 0$$

[Not a coincidence!]

i.e. integral of g over <u>$\partial\Omega$ must vanish</u>!

↑ also 1-D if $\partial\Omega$ conn.

(Let $\langle g \rangle = \int g \, dS / \int 1 \, dS$

Then $\int_{\partial\Omega} (g - \langle g \rangle) dS = 0$)

1.5 Green's theorems

Now assume Ω as above, $u, v \in C^2(\Omega) \cap C^1(\bar\Omega)$.

Then

$$\int_\Omega \frac{\partial^2 u}{\partial x_k^2} v \, dx = \int_{\partial\Omega} n_k \frac{\partial u}{\partial x_k} v \, dS - \int_\Omega \frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k} dx.$$

Summing over k we get

$$\int_\Omega \Delta u \, v \, dx = \int_{\partial\Omega} \underbrace{\langle n, \nabla u\rangle}_{=\partial_n u} v \, dS - \int_\Omega \langle \nabla u, \nabla v\rangle dx$$

i.e. 1^{st} Green's formula. Changing roles of u & v we also get

$$\int_\Omega u \Delta v \, dx = \int_{\partial\Omega} u \, \partial_n v \, dS - \int_\Omega \langle \nabla u, \nabla v\rangle dx,$$

and substracting these we get the 2^{nd} Green's formula

$$\int_\Omega \Delta u \cdot v - u \Delta v \, dx = \int_{\partial\Omega} \partial_n u \cdot v - u \partial_n v \, dS.$$

So - summing up here are the four important formulas:

$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} \langle n, F\rangle dS \quad \text{(Divergence thm.)}$$

$$\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \partial_n u \, dS \quad \text{(Gauss thm.)}$$

$$\int_\Omega \Delta u \, v \, dx = \int_{\partial\Omega} \partial_n u \cdot v \, dS - \int_\Omega \langle \nabla u, \nabla v\rangle dx \quad (1^{st} \text{ Green})$$

$$\int_\Omega \Delta u \cdot v - u \Delta v \, dx = \int_{\partial\Omega} \partial_n u \cdot v - u \partial_n v \, dS \quad (2^{nd} \text{ Green})$$

1.6. Representation thms.

Recall that the radial solutions of $\Delta u = 0$ are

$$u(x) = \begin{cases} C \ln|x|, & d = 2, \quad x \neq 0 \\ C|x|^{2-n}, & d = 3, 4, \ldots, \quad x \neq 0 \end{cases}$$

[HW 2 / Ex 2

So note that there are not def. at 0]

Def. The functions

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln|x|, & x \neq 0, \ d = 2, \\ \frac{1}{(d-2)\omega_d} |x|^{2-d}, & x \neq 0, \ d = 3, 4, \ldots \end{cases}$$

where ω_d = area of $\{|x| = 1;\ x \in \mathbb{R}^d\}$

$$\left(= d\pi^{d/2} / \Gamma\left(\tfrac{d}{2} + 1\right) \quad \text{Euler's Gamm fnct} \right)$$

are fundamental solutions of Δ in $\mathbb{R}^d$. Note

$$\omega_3 = 3\pi^{3/2} / \Gamma(5/2) = 3\pi^{3/2} / \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right)$$

$$= 2\pi^{3/2} / \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \pi^{3/2} / \Gamma\left(\tfrac{1}{2}\right) = 4\pi^{3/2} / \pi^{1/2} = 4\pi$$

i.e.

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln|x|, & x \neq 0, \ d = 2 \\ \frac{1}{4\pi} |x|^{-1}, & x \neq 0, \ d = 3 \end{cases}$$

The important cases!

The normalizations chosen are important for the following reason:

Thm. (Rep. thm.) If Ω is a bnd C^1-domain of $\mathbb{R}^d, d \geq 2$ and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, then $\forall\, x \in \Omega$:

$$u(x) = \int_{\partial\Omega} \left\{ \Phi(x-y) \frac{\partial u(y)}{\partial n(y)} - \frac{\partial \Phi(x-y)}{\partial n(y)} u(y) \right\} dS(y)$$

$$- \int_{\Omega} \Phi(x-y)\, \Delta u(y)\, dy, \quad x \in \Omega$$

Note that this is true for any function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$.
If $\Delta u = 0$ in Ω we get a corollary:

Cor. (Rep. thm. for harmonic functions) If Ω is as above, and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, and harmonic in Ω i.e.

$$\Delta u = 0 \quad \text{in } \Omega,$$

then $\forall\, x \in \Omega$:

$$u(x) = \int_{\partial\Omega} \left\{ \Phi(x-y) \frac{\partial u(y)}{\partial n(y)} - \frac{\partial \Phi(x-y)}{\partial n(y)} u(y) \right\} dS(y).$$

Hence: If one knows u & $\partial u/\partial n$ on $\partial\Omega$, one knows explicitly u inside!

Is this useful? Not immediately. Why?

Well, you can't fix both $u|_{\partial\Omega}$ & $\partial_n u|_{\partial\Omega}$ for sols of harmonic functions!

Explanation Let $f \in C(\partial\Omega)$. The problem

$$(0) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega \end{cases}$$

has at most one solution (We don't know yet if it even has a solution!).

If $g \neq \partial u/\partial n$ (u unique sol of (0)), then there is no v s.t.

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v|_{\partial\Omega} = f, \ \partial_n v|_{\partial\Omega} = g \end{cases}$$

and in particular if

$$w(x) := \int_{\partial\Omega} \left\{ \Phi(x-y)\, g(y) - \frac{\partial \Phi(x-y)}{\partial n(y)} f(y) \right\} dS(y),$$

either $w|_{\partial\Omega} \neq f$ or $\partial_n w \neq g$ (if these bvalues even exist!)

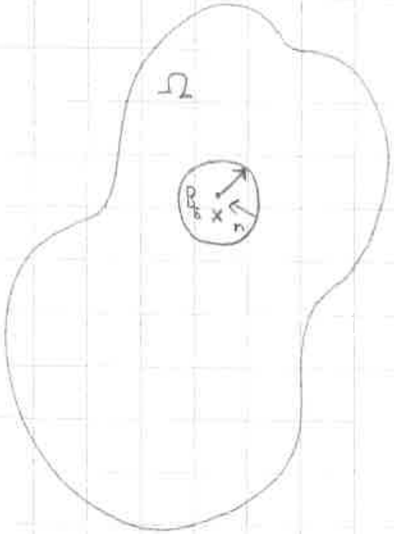
Proof of Rep. thm: Let $x \in \Omega$ be fixed and choose $\varepsilon > 0$ so small that

$$\overline{B_\varepsilon(x)} \subset \Omega$$

Let $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(x)}$.

Note that Ω_ε is a C^1-domain, and for the exterior unit normal we have

$$n = \begin{cases} n = \text{ext. unit normal of } \Omega \text{ on } \partial\Omega \\ \dfrac{x-y}{|x-y|} \quad \text{on } \partial B_\varepsilon(x) \end{cases}$$



Now

$$y \mapsto \Phi(x-y)$$

is harmonic in Ω_ε since $y \neq x$ in Ω_ε! Hence Green =>

$$\int_{\Omega_\varepsilon} \underbrace{\Delta\Phi(x-y)\cdot u(y)}_{=0} - \Phi(x-y)\Delta u(y)\,dy$$

$$= \int_{\partial\Omega_\varepsilon} \frac{\partial\Phi(x-y)}{\partial n(y)} u(y) - \Phi(x-y)\frac{\partial u(y)}{\partial n(y)}\,dS(y)$$

i.e we have

$$(☺)_\varepsilon$$

$$-\int_{\Omega_\varepsilon} \Phi(x-y)\Delta u(y)\,dy = \int_{\partial\Omega} + \int_{\partial B_\varepsilon} \frac{\partial\Phi(x-y)}{\partial n(y)} u(y) - \Phi(x-y)\frac{\partial u(y)}{\partial n(y)}\,dS(y).$$

Now Dom. Conv. (Φ integrable$^{\text{loc.}}$ over $\mathbb{R}^d$)

$$\lim_{\varepsilon\to+0} \int_{\Omega_\varepsilon} \Phi(x-y)\Delta u(y)\,dy = \int_{\Omega} \Phi(x-y)\Delta u(y)\,dy$$

Exc. the integral on Ω we want

The integral over $\partial\Omega$ is exactly what it should.

Let's study $\int_{\partial B_\varepsilon} \dots dS$.

I. Let's 1st look int

$$\int_{\partial B_\varepsilon(x)} \Phi(x-y)\frac{\partial u(y)}{\partial n(y)}\,dS(y).$$

Now $\forall$ $|y-x| = \varepsilon$,

$$|\Phi(x-y)| = \begin{cases} C \ln\varepsilon \\ C\varepsilon^{2-d} \end{cases}$$

and area of $\partial B_\varepsilon(x) = \omega_d \varepsilon^{d-1}$; hence since $\frac{\partial u}{\partial n}$ unif. bnd,

$$\left|\int_{\partial B_\varepsilon(x)} \Phi(x-y)\frac{\partial u(y)}{\partial n(y)}\,dS(y)\right| \leq \begin{cases} C\varepsilon \ln\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to+0 \\ \quad \text{if } d=2 \\ C\varepsilon^{d-1}\cdot\varepsilon^{2-d} = C\varepsilon \longrightarrow 0 \text{ as } \varepsilon\to 0 \\ \quad \text{if } d\geq 3 \end{cases}$$

$$\therefore \lim_{\varepsilon\to+0}\int_{\partial B_\varepsilon}\Phi(x-y)\frac{\partial u(y)}{\partial n(y)}\,dS(y) = 0$$

II We are still missing the term $u(x)$, and so it has to come from

$$\int_{\partial B_\varepsilon(x)} \frac{\partial\Phi(x-y)}{\partial n(y)}\,u(y)\,dS(y).$$

Let's see why! We need to compute $\partial\Phi(x-y)/\partial n(y)$.
Now

$$\nabla_y\Phi(x-y) = \begin{cases} -\frac{1}{2\pi}\frac{x-y}{|x-y|^2}(-1) = \frac{1}{2\pi}\frac{x-y}{|x-y|^2}, & d=2 \\ \frac{1}{(d-2)\omega_d}\cdot(2-d)\,r^{1-d}\underbrace{\frac{x-y}{|x-y|}}_{=r}(-1), & d\geq 3 \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi}\frac{x-y}{|x-y|}, & d=2 \\ \frac{1}{\omega_d}\frac{x-y}{|x-y|}\,r^{1-d}, & d\geq 3. \end{cases}$$

Also on $\partial B_\varepsilon(x)$ the ext. unit normal of Ω_ε is $\frac{y-x}{|x-y|}$, and hence if $d=2$, $x\in\partial B_\varepsilon(x)$ ($\Leftrightarrow |x-y|=\varepsilon$)

$$\frac{\partial\Phi(x-y)}{\partial n(y)} = \left\langle\frac{y-x}{|x-y|}, \frac{1}{2\pi}\frac{x-y}{|x-y|^2}\right\rangle = -\frac{1}{2\pi\varepsilon}$$

and if $d\geq 3$, $|x-y|=\varepsilon$,

$$\frac{\partial\Phi(x-y)}{\partial n(y)} = \left\langle\frac{y-x}{|x-y|}, \frac{1}{\omega_d}\frac{x-y}{|x-y|}r^{1-d}\right\rangle = -\frac{1}{\omega_d}\varepsilon^{1-d}$$

i.e. always

$$\frac{\partial\Phi(x-y)}{\partial n(y)} = -\frac{1}{\omega_d}\varepsilon^{1-d},$$

and we have

$$\int_{\partial B_\varepsilon(x)}\frac{\partial\Phi(x-y)}{\partial n(y)}u(y)\,dS(y) = -\frac{1}{\omega_d}\varepsilon^{1-d}\int_{|y-x|=\varepsilon}u(y)\,dS(y)$$

$$= -\frac{1}{\cancel{\omega_d}}\varepsilon^{1-d}\varepsilon^{d-1}\cancel{\omega_d}\frac{1}{|\partial B_\varepsilon(x)|}\int_{|y-x|=\varepsilon}u\,dS(y)$$

$$\xrightarrow[\varepsilon\to+0]{} -u(x).$$

Hence taking $\varepsilon\to+0$ in $(\ ")_\varepsilon$ we get

$$-\int_\Omega\Phi(x-y)\,\Delta u(y)\,dy = \int_{\partial\Omega}\frac{\partial\Phi(x-y)}{\partial n(y)}u(y) - \Phi(x-y)\frac{\partial u(y)}{\partial n(y)}\,dS(y) + u(x). \quad\square$$

1.7. Green's functions.

Assume Ω a bnd C^1-domain, $u \in C^2(\Omega) \cap C^1(\bar\Omega)$ harmonic in Ω. Then by Rep.-thm:

$$(1) \quad u(x) = \int_{\partial\Omega} \Phi(x-y)\frac{\partial u(y)}{\partial n(y)} - \frac{\partial \Phi(x-y)}{\partial n(y)} u(y)\, dS(y).$$

Assume now $v \in C^2(\Omega) \cap C^1(\bar\Omega)$ is also harmonic in Ω: then

$$(2) \quad 0 = \int_{\partial\Omega} v(y)\frac{\partial u(y)}{\partial n(y)} - \frac{\partial v(y)}{\partial n(y)} u(y)\, dS(y)$$

$$\overset{\text{Green } 2^{nd}}{=} \Big(\int_\Omega v \underbrace{\Delta u}_{=0} - \underbrace{\Delta v}_{=0} \cdot u\, dy$$

Substracting (2) from (1) we get

$$u(x) = \int_{\partial\Omega} \{\Phi(x-y) - v(y)\}\frac{\partial u(y)}{\partial n(y)} - \frac{\partial\{\Phi(x-y) - v(y)\}}{\partial n(y)} u(y)\, dS(y)$$

|| i.e. we may add any harmonic fnc to Φ and Rep. thm still holds ! ||

Why is this useful? Well, given a fixed x assume that we can find harmonic v_x s.t.

i.e. $\Delta_y v_x(y) = 0$ in Ω

s.t. $v_x(y) = \Phi(x-y)$ $\forall\, y \in \partial\Omega$. Then letting

$$G(x,y) := \Phi(x-y) - v_x(y)$$ [Dirichlet Green's function for Ω]

we have

depends only on Dirichlet data ?

$$(*) \quad u(x) = -\int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)\, u(y)\, dS(y) \ !$$

Does this solve all our Dirichlet problems? Well there are problems:

(a) How to find G i.e. $v_x(y)$ for given $x \in \Omega$?
This is equivalent to solving another Dirichlet problem (actually one for every $x \in \Omega$):

$$\begin{cases} \Delta_y v_x(y) = 0 & \forall\, y \in \Omega \\ v_x = \Phi(x - \cdot) & \text{on } \partial\Omega. \end{cases}$$

Not a big help!

(b) Assume we want to solve

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$$

Is $u(x) = -\int_{\partial\Omega} \partial_n G(x-y) f(y)\, dS(y)$ a solution?

Note that it is not obvious that $\Delta_x G(x-y) = 0$ and since $|\Phi(x-y)| \to \infty$ as $x \to y$, it is not obvious how to prove that

$$\lim_{\substack{\Omega \ni x \to x_0 \\ x_0 \in \partial\Omega}} u(x) = f(x_0).$$

1.7. Existence of G

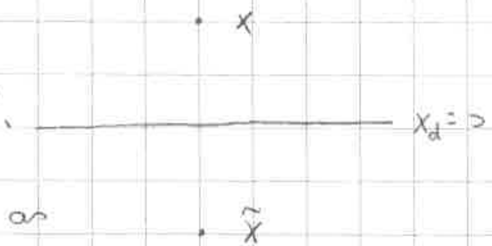
This usually works only for domains with enough symmetries. One can prove that G exists in the general case, but there is usually no explicit formula! However sometimes this works.

Example: if $\Omega = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d;\ x_d > 0\}$ even if not bnd.

Let $x \in \Omega$, $x_d > 0$.

Let $\tilde{x} = (x_1, \dots, x_{d-1}, -x_d)$

"reflection of x w.r.t. $x_d = 0$".



Then $\Phi(\tilde{x} - y)$ is harmonic as a function of y if $y \neq \tilde{x}$ i.e.

$$\Delta_y \Phi(\tilde{x} - y) = 0 \text{ in } \Omega.$$

Also if $x_d = 0 \Rightarrow x = \tilde{x} \Rightarrow \Phi(x-y) = \Phi(\tilde{x} - y)$.

Hence we may take (think why - !)

$$G(x-y) = \left(\Phi(x-y) - \Phi(\tilde{x}-y)\right)$$

Computing this expl. out is left to HW.

One can do in principle the same thing in $B_R(0)$:

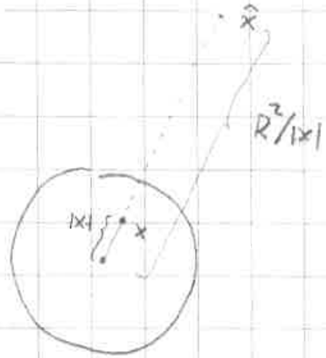
Given $x \in B_R(0)$, let

$$\tilde{x} = \frac{R^2}{|x|^2} x,$$

$$|\tilde{x}| = R^2/|x| = R\left(\frac{R}{|x|}\right) > R$$

$\to \infty$ as $|x| \to 0$

$\to R$ as $|x| \to R$



This is the Kelvin-transform of x

Then we define again

$$G(x,y) = \Phi(x-y) - \left(\frac{R}{|x|}\right)^{d-2} \Phi(\tilde{x}, y).$$

You will show in HW 4 that for $|y| = R$, $|x| < R$

$$-\frac{\partial G(x,y)}{\partial n(y)} = \frac{1}{R\omega_d} \frac{R^2 - |x|^2}{|x-y|^d},$$

and hence our guess for the solution of

$$\Delta u = 0 \text{ in } B_R(0), \quad u(x) = f(x) \text{ if } |x| = R$$

would be

$$u(x) = \frac{1}{R\omega_d} \int_{|y|=R} \frac{R^2-|x|^2}{|x-y|^d} f(y)\, dS(y).$$

Now one can directly compute that

$$\Delta_x \left(\frac{R^2-|x|^2}{|x-y|^d} \right) = 0 \quad \forall\ |x|<R,\ |y|=R.$$

So $\Delta u = 0$ in $B_R(0)$. One also has:

Prop. $\lim_{B_R(0) \ni x \to x_0} u(x) = f(x_0)$ for all $x_0 \in \partial B_R(0)$.

Pf. First an important observation: Since constant 1 is harmonic, we have (use (*) with $u \equiv 1$)

$$(**) \qquad 1 = \frac{1}{R\omega_d} \int_{|y|=R} \frac{R^2-|x|^2}{|x-y|^d}\, dS(y).$$

Fix $x_0 \in \partial B_R(0)$. Let $\varepsilon > 0$ and choose $\delta > 0$ s.t.

$$y \in \partial B_R(0),\ |y-x_0| < \delta \Rightarrow |f(y) - f(x_0)| < \varepsilon.$$

Now $\Rightarrow$

$$\varphi(x_0) = \frac{1}{R\omega_d} \int_{|y|=R} \frac{R^2-|x|^2}{|x-y|^d} \varphi(x_0)\, dS(y)$$

$$\Rightarrow \quad u(x) - u(x_0) = \frac{1}{R\omega_d} \int_{|y|=R} \frac{R^2-|x|^2}{|x-y|^d} \left(f(y) - f(x_0)\right) dS(y)$$

$$= \frac{1}{R\omega_d} \left(\int_{\substack{|y|=R \\ |y-x_0| \geq \delta}} + \int_{\substack{|y|=R \\ |y-x_0|<\delta}} \right) \frac{R^2-|x|^2}{|x-y|^d} \left(f(y) - f(x_0)\right) dS(y)$$

$$=: I_1(\delta) + I_2(\delta).$$

Now when $|x - x_0| < \delta/2$, we have

$$|y-x| \geq \big| |y-x_0| - |x_0 - x| \big| > \delta/2$$

and

$$\lim_{x \to x_0} \frac{1}{R\omega_d} \int_{\substack{|y|=R \\ |y-x_0| \geq \delta}} \frac{R^2-|x|^2}{|x-y|^d} \left(f(y) - f(x_0)\right) dS(y) = 0 \quad (R^2 - |x|^2 \to 0!).$$

On the other hand,

$$|I_2(\delta)| \leq \frac{1}{R\omega_d} \int_{\substack{|y|=R \\ |y-x_0|<\delta}} \overbrace{\frac{R^2-|x|^2}{|x-y|^d}}^{>0} \underbrace{|f(y) - f(x_0)|}_{<\varepsilon} dS(y)$$

$$< \varepsilon \frac{1}{R\omega_d} \int_{\substack{|y|=R \\ |y-x_0|<\delta}} \frac{R^2-|x|^2}{|x-y|^d}\, dS(y) \leq \varepsilon \frac{1}{R\omega_d} \int_{|y|=R} \frac{R^2-|x|^2}{|x-y|^d}\, dS(y)$$

$$= \varepsilon.$$

Since $\varepsilon > 0$ was arb., we have proven the claim! $\square$

2. Subharmonic functions and the Mean Value Property

2.1. Subharmonic functions - 1st def.

Let $\Omega \subset \mathbb{R}^d$ be open. A $u \in C^2(\Omega)$ is <u>subharmonic</u> if

(Sb) $\qquad \Delta u \geq 0 \quad$ in Ω.

A function $u \in C^2(\Omega)$ is <u>superharmonic</u> if $-u$ is subharm., i.e.

(Su) $\qquad \Delta u \leq 0 \quad$ in Ω

We've seen in HW1 that $\Delta u > 0$ in $\Omega \Rightarrow u$ has no local maxima
$\Delta u < 0$ — " — — " — minima.

To extend this to harmonic case $\Delta u = 0$ one needs information of mean values of sub & superharmonic functions.
Also these are needed in an ingenious pf. of solvability of Dirichlet-problems (not needing an <u>explicit</u> Green's function) due to O. Perron (1923)

2.2. Mean Value Property

Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ & Ω a bnd C^1-domain.

As before

$$u(x) = \int_{\partial\Omega} \Phi(x-y)\frac{\partial u}{\partial n(y)} - \frac{\partial \Phi(x-y)}{\partial n(y)} u(y)\, dS(y) - \int_{\Omega} \Phi(x-y)\Delta u(y)\, dy$$

and if v harmonic, 2nd Green $\Rightarrow$

$$0 = \int_{\Omega} v(y)\frac{\partial u}{\partial n(y)} - \frac{\partial v(y)}{\partial n(y)} u(y)\, dS(y) - \int_{\Omega} v(y)\Delta u(y)\, dy.$$

Hence we have for the Dir. Green's function (if it exists)

$$u(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial n(y)}(x,y)\, u(y)\, dS(y) - \int_{\Omega} G(x,y)\Delta u(y)\, dy.$$

Let's choose $\Omega = B_R(x_0)$, $x = x_0$. Then

$$u(x_0) = \frac{1}{R\omega_d}\int_{|y-x_0|=R} \frac{R^2-|x-x_0|^2}{|y-x|^d}\Bigg|_{x=x_0} u(y)\, dS(y)$$

$$\int_{|y-x_0|<R} G(x,y)\Bigg|_{x=x_0} \Delta u(y)\, dy.$$

$$= \frac{R^{1-d}}{\omega_d}\int_{|y-x_0|=R} u(y)\, dS(y) - \int_{|y-x_0|<R} G(x_0,y)\Delta u(y)\, dy$$

What can we read from this?

Note that $R^{1-d}/\omega_d = 1/R^{d-1}\omega_d = 1/m(\partial B_R)$, and hence

$$\frac{R^{1-d}}{\omega_d}\int_{|y-x_0|=R} u(y)\,dS(y) = \frac{1}{m(\partial B_R)}\int_{\partial B_R(x_0)} u\,dS(y)$$

$$= \fint_{\partial B_R(x_0)} u\,dS(y) \quad = \text{average/mean value of } u \text{ over } \partial B_R(x_0).$$

Hence, since $G(x_0,y)\geq 0$ (Check this yourself!)

$$u(x_0) = \fint_{\partial B_R(x_0)} u\,dS(y) \underbrace{- \int_{B_R(x_0)} G(x_0,y)\,\Delta u(y)\,dy}_{\substack{\leq 0,\ \text{if } \Delta u \geq 0 \\ \geq 0,\ \text{if } \Delta u \leq 0}}$$

Hence we have shown:

Prop. a) If $u\in C^2(\Omega)$ is subharmonic in Ω i.e. $\Delta u\leq 0$, then $\forall\ \overline{B_R(x_0)}\subset\Omega$ we have

$$u(x_0)\leq \fint_{\partial B_R(x_0)} u\,dS(y)$$

b) If $u\in C^2(\Omega)$ is superharmonic in Ω i.e. $\Delta u\geq 0$ in Ω, then $\forall\ \overline{B_R(x_0)}\subset\Omega$ we have

$$u(x_0)\geq \fint_{\partial B_R(x_0)} u\,dS(y).$$

c) If $u\in C^2(\Omega)$ is harmonic in Ω, then

$$u(x_0) = \fint_{\partial B_R(x_0)} u\,dS(y),$$

So, u harmonic $\Rightarrow$ $u(x_0)$ = mean value of u over boundaries of balls centered at x_0

subharmonic $\Rightarrow$ $u(x_0)\leq$ — " —

superharmonic $\Rightarrow$ $u(x_0)\geq$ — " —

2.2. Generalization of sub/superharmonicity

Note that the mean value of u is well defined if integrals over $\partial B_R(x_0)$'s exist. Hence we may drop the requirement that $u\in C^2(\Omega)$ and just define

Def. $\{u\in C(\Omega)\}$ $u:\Omega\to\mathbb{R}$ is subharmonic if $\forall\ x_0, R$ s.t. $\overline{B_R(x_0)}\subset\Omega$ we have

$$u(x_0)\leq \fint_{\partial B_R(x_0)} u\,dS(y).$$

Similarly, $u\in C(\Omega)$ is superharmonic if $\forall\ x_0, R$ s.t. $\overline{B_R(x_0)}\subset\Omega$ we have

Of course this raises the question: how to define that u is harmonic if u a priori is only continuous. We will see in a moment that all these defs are in fact equivalent.

<u>Lemma</u> $\Omega \subset \mathbb{R}^d$, $d \geq 2$ bnd domain; if $u \in C(\bar{\Omega})$ is subharmonic in E, then either

- u is constant in Ω

or

- $u(x) < \sup_{\partial\Omega} u \quad \forall x \in \Omega$.

This is called <u>strong maximum principle</u>. Note that superharmonic functions satisfy a corresponding <u>strong minimum principle</u>.

Pf. $u \in C(\bar{\Omega})$ & Ω bnd $\Rightarrow \exists x_0 \in \bar{\Omega}$ s.t.

$$u(x_0) = \sup_{\bar{\Omega}} u .$$

Assume $u \not\equiv u(x_0)$ (i.e. u not constant). If $x_0 \in \Omega$, then $\forall R > 0$ s.t. $\overline{B_R(x_0)} \subset \Omega$,

$$\sup_{\bar{\Omega}} = u(x_0) = \fint_{\partial B_R(x_0)} u \, dS$$

$$\Rightarrow u \equiv u(x_0) \text{ in } \overline{B_R(x_0)} .$$

Let $M = \{ x \in \Omega ; \ u(x) = u(x_0) \}$.
Previous arg $\Rightarrow M$ open.
u cont. $\Rightarrow M$ closed.
Ω connected $\Rightarrow M$ is either $= \Omega$ or $\emptyset$.
If $M = \Omega$, then u constant; hence $M = \emptyset$ and

$$u(x) < \sup_{\bar{\Omega}} u \quad \forall x \in \Omega$$

and thus $\sup_{\bar{\Omega}} u = \max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. $\square$

<u>Def.</u> $u \in C(\Omega)$ satisfies the <u>mean value property</u> (MVP) if

$$u(x_0) = \fint_{\partial B_R(x_0)} u \, dS \quad \forall x_0 \in \Omega, R > 0 \text{ s.t. } \overline{B_R(x_0)} \subset \Omega .$$

Note: $u \in C^2(\Omega)$ & $\Delta u = 0$ in $\Omega \Rightarrow u$ sat. (MVP)!

Now we have:

<u>Prop.</u> $\Omega \subset \mathbb{R}^d$, $d \geq 2$ bnd domain. If u sat. (MVP) in Ω, then either u constant in Ω, or

$$|u(x)| < \sup_{\partial\Omega} |u| \quad \forall x \in \Omega .$$

Pf. u & $-u$ are subharmonic. $\square$

(In particular, (MVP) $\Rightarrow \max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$)

(slightly stronger version)

We can now give another pf of the uniqueness of the Dir. problem when Ω bnd domain:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega \end{cases}, \quad u \in C^2(\Omega) \cap C(\bar\Omega)$$

Namely, if u_1 & u_2 are two sols, then $v = u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v|_{\partial\Omega} = (u_1 - u_2)|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

hence $\max_{\Omega} |v| = \max_{\partial\Omega} |v| = 0 \Rightarrow v = 0$ in Ω.

Finally, one can characterize harmonicity with (MVP):

<u>Thm.</u> The following are equiv.; let $\Omega \Subset \mathbb{R}^d$.

(a) $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω

(b) $u \in C(\Omega)$ and u satisfies (MVP).

<u>Remark:</u> Note that (b) assumes <u>only</u> the continuity of u.

Pf. (a) $\Rightarrow$ (b) already done.

So, assume that $u \in C(\Omega)$ and (MVP) holds.
Fix $x_0 \in \Omega$ and $R > 0$ s.t. $\overline{B_R(x_0)} \subset \Omega$.

Using Poisson kernel, we can find $v \in C^2(B_R(x_0)) \cap C(\overline{B_R(x_0)})$ s.t.

$$\begin{cases} \Delta v = 0 & \text{in } B_R(x_0) \\ v = u & \text{on } \partial B_R(x_0) \end{cases}$$

Now u sat. (MVP), v sat. (MVP) since v harmonic
$\Rightarrow w = u - v$ sat. (MVP) in $B_R(x_0)$.

Hence Max. Princ. $\Rightarrow \sup_{B_R(x_0)} |w| = \sup_{\partial B_R(x_0)} |w| = 0$ i.e.

$u = v$ in $B_R(x_0)$ $\Rightarrow$ claim. $\square$

We will return to sub (& super) harmonic functions bit later.

2.3. <u>Harnack's inequality</u>

<u>Thm.</u> (Harnack 1889) $\Omega \Subset \mathbb{R}^d$, $\Delta u = 0$ in Ω and <u>$u \geq 0$</u>. Then

$\forall x \in \overline{B_\rho(x_0)} \subset B_R(x_0) \subset \Omega$

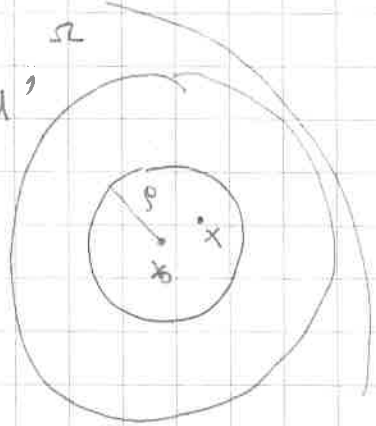
we have the inequality

$$\left(\frac{R}{R+\rho}\right)^{d-2}\frac{R-\rho}{R+\rho}\,u(x_0) \le u(x) \le \left(\frac{R}{R-\rho}\right)^{d-2}\frac{R+\rho}{R-\rho}\,u(x_0)$$

"We can compare $u(x)$ to $u(x_0)$".

"If $u(x_0)$ small $\rightsquigarrow$ $u(x)$ pretty small"

" $u(x_0)$ large $\rightsquigarrow$ $u(x)$ " – large".



Pf. We may make a translation s.t. $x_0 = 0$. Then

$$\underset{\text{Poisson kernel}}{u(x) =} \frac{R^2-|x|^2}{R\omega_d}\int_{\partial B_R}\frac{u(y)}{|x-y|^d}\,dS \overset{(\Delta\text{ ineq})}{\le} \frac{R^2-|x|^2}{R\omega_d}\int_{\partial B_R}\frac{u(y)}{(\underbrace{|y|}_{=R}-|x|)^d}\,dS(y)$$

$$= \frac{\overbrace{R^2-|x|^2}^{(R-|x|)(R+|x|)}}{(R-|x|)^d}\,\frac{1}{R\omega_d}\underbrace{\int_{\partial B_R} u(y)\,dS(y)}_{\overset{(MVP)}{=} R^{d-1}\omega_d u(0)} = \left(\frac{R}{R-|x|}\right)^{d-2}\frac{R+|x|}{R-|x|}\,u(0)$$

Now $|x|<\rho \Rightarrow$

$$u(x) \le \left(\frac{R}{R-\rho}\right)^{d-2}\frac{R+\rho}{R-\rho}\,u(0).$$

The other direction we leave as (HW). □

This has a dramatic consequence:

Thm. (Liouville's thm) Assume $u \ge 0$ and $\Delta u = 0$ in $\mathbb{R}^d$. Then u is constant.

Pf. Fix $x_0 \in \mathbb{R}^d$, $\rho > 0$. Then Harnack $\Rightarrow$

$$u(x) \le \left(\frac{R}{R-\rho}\right)^{d-2}\frac{R+\rho}{R-\rho}\,u(x_0) \qquad \forall R>\rho$$

Letting $R \to \infty$ we get

$$u(x) \le u(x_0).$$

Similarly the lower bound in Harnack gives

$$u(x_0) \le u(x).$$

Hence $u(x) = u(x_0)$ $\forall x \in \mathbb{R}^d$. □
in $\mathbb{R}^d$

Cor. If u harmonic & bdd. from below by a constant $\Rightarrow$ $u = $ const.

Pf. Choose $k \in \mathbb{R}$ s.t. $u(x) \ge k$ $\forall x \in \mathbb{R}^d$. Then $v := u - k$ is harmonic, $v \ge 0$, in $\mathbb{R}^d$

$\Rightarrow$ v constant $\Leftrightarrow$ u constant. □

2.4. Basic Properties of sub- and superharmonic functions

In the proof of the solvability of the Dirichlet-problem we need several further properties of sub- and superharmonic functions.

Def. If $\Omega \Subset \mathbb{R}^d$, let

$$\mathcal{S}(\Omega) := \{ v \in C(\Omega);\ v \text{ subharmonic in } \Omega \}$$
$$\Sigma(\Omega) := \{ \ -\text{"}-\ ;\ \text{" superharmonic "} \}$$

We have:

Prop. 2.4.1. If $v \in \begin{cases} \Sigma(\Omega) \\ \mathcal{S}(\Omega) \end{cases}$, and $\Omega' \Subset \Omega$ then $v|_{\Omega'} \in \begin{cases} \Sigma(\Omega') \\ \mathcal{S}(\Omega') \end{cases}$.

Pf. Trivial □

Prop. 2.4.2 If $v_i \in \mathcal{S}(\Omega)$ (or $\in \Sigma(\Omega)$), $i = 1, \dots, k$, then

$$v_1 + \dots + v_k \in \mathcal{S}(\Omega) \quad (\text{or } \Sigma(\Omega)).$$

Pf. $v_1 + \dots + v_k \in C(\Omega)$ and for all $\overline{B_R(x_0)} \subset \Omega$,

$$v_1(x_0) + \dots + v_k(x_0) \leq \fint_{\partial B_R(x_0)} v_1 \, dS + \dots + \fint_{\partial B_R(x_0)} v_k \, dS$$
$$= \fint_{\partial B_R(x_0)} (v_1 + \dots + v_k) \, dS.$$

Similarly for superharmonicity. □

Prop. 2.4.2. If $v_1, \dots, v_k \in \mathcal{S}(\Omega)$, then

$$\max\{v_1, \dots, v_k\} \in \mathcal{S}(\Omega).$$

Pf. $\forall\ \overline{B_R(x_0)} \subset \Omega$ we have

$$v_i(x_0) \leq \fint_{\partial B_R(x_0)} v_i \, dS \leq \fint_{\partial B_R(x_0)} \max\{v_1, \dots, v_k\} \, dS$$

$$\Rightarrow \max\{v_1(x_0), \dots, v_k(x_0)\} \leq \fint_{\partial B_R(x_0)} \max\{v_1, \dots, v_k\} \, dS. \ \square$$

Prop. 2.4.2. If $f : \mathbb{R} \to \mathbb{R}$ is convex & non-decreasing, then $v \in \mathcal{S}(\Omega) \Rightarrow f \circ v \in \mathcal{S}(\Omega)$.

Pf. $\begin{cases} f \text{ convex \& non-dec.} \\ \text{Jensen's ineq.} \end{cases} \Rightarrow \forall\ \overline{B_R(x)} \subset \Omega$ we have

$$f(v(x_0)) \leq f\Big(\fint_{\partial B_R(x_0)} v \, dS \Big) \overset{\text{Jensen}}{\leq} \fint_{\partial B_R(x_0)} (f \circ v) \, dS. \ \square$$

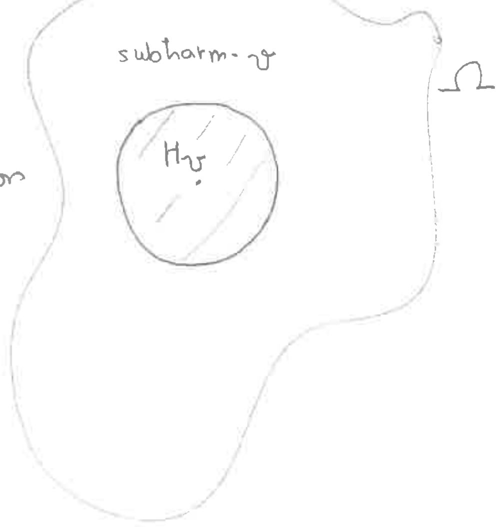
There is a very important construction that creates subharmonic modifications of harmonic functions:

Assume $v \in \mathcal{S}(\Omega)$, $\Omega \Subset \mathbb{R}^d$ open.

Fix $B_\rho(\xi) \subset \Omega$ s.t. $\overline{B_\rho(\xi)} \subset \Omega$.

Solve using the Poisson kernel the Dirichlet problem

$$\Delta H = 0 \text{ in } B_\rho(\xi), \quad H|_{\partial B_\rho(\xi)} = v$$



[H_v is the unique harmonic extension of v to $B_\rho(\xi)$]

Define

$$v_{\rho,\xi} = \begin{cases} v & \text{in } \Omega \setminus B_\rho(\xi) \\ H_v & \text{in } B_\rho(\xi) \end{cases}$$

Then $v_{\rho,\xi}$ is continuous in Ω but usually not C^1!

Now we have

Prop. (a) $v \le v_{\rho,\xi}$ in Ω

(b) $v_{\rho,\xi} \in \sigma(\Omega)$.

Pf. (a) Now $v \in \sigma(B_\rho(\xi))$ and H_v harmonic

$\Rightarrow$ $v - H_v \in \sigma(B_\rho(\xi))$. Also $v - H_v|_{\partial B_\rho(\xi)} = 0$

Max. Princ. $\Rightarrow$ $v - H_v \le 0$ in $B_\rho(\xi)$ $\Leftrightarrow$ $v \le H_v$ in $B_\rho(\xi)$.

(b) Let $\overline{B_R(x_0)} \subset \Omega$.

If $x_0 \in \Omega \setminus B_\rho(\xi)$, then

$$v_{\rho,\xi}(x_0) = v(x_0) \le \fint_{\partial B_R(x_0)} v \, dS \le \fint_{\partial B_R(x_0)} v_{\rho,\xi} \, dS.$$

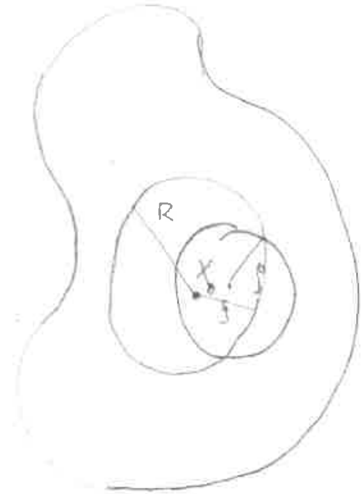
Assume then $x_0 \in B_\rho(\xi)$.

Let's make a counter assumption: for some R, s.t. $\overline{B_R(x_0)} \subset \Omega$ we have

$$v_{\rho,\xi}(x_0) > \fint_{\partial B_R(x_0)} v_{\rho,\xi} \, dS.$$

Let

$$w = \begin{cases} v_{\xi,\rho} & \text{in } \Omega \setminus B_R(x_0) \\ H_{v_{\xi,\rho}} & \text{in } B_R(x_0) \end{cases}$$



We claim that

(*) $w \ge v_{x_0,R}$ in Ω.

Now in $\Omega \setminus B_R(x_0)$,

$$w = v_{\xi,\rho} \ge v,$$

and on $\partial B_R(x_0)$,

$$w = H_{v_{\xi,\rho}}|_{\partial B_R} = v_{\xi,\rho}|_{\partial B_R(x_0)} \ge v|_{\partial B_R(x_0)} = v_{x_0,R}|_{\partial B_R}$$

and max. princ. gives (*).

w sat. (MVP) in $B_R(x_0)$,

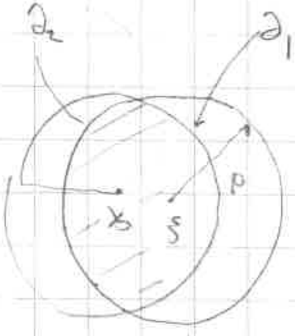
Counter asspt.

(*) $$w(x_0) = \fint_{\partial B_R(x_0)} w \, dS = \fint_{\partial B_R(x_0)} v_{\rho,\xi} \, dS < v_{\rho,\xi}(x_0)$$

Now $v_{\rho,\xi} - w = H_v - H_{v_{\rho,\xi}}$ in $B_\rho(\xi) \cap B_R(x_0)$

(**) i.e. $\Delta(v_{\rho,\xi} - w) = 0$ in $B_\rho(\xi) \cap B_R(x_0)$

$$\partial(B_\rho(\xi) \cap B_R(x_0)) = \partial_1 \cup \partial_2$$



(*), (**) $\Rightarrow$ $v_{\rho,\xi} - w$ must have a pos. maximum on $\partial_1 \cup \partial_2$.

$$(v_{\rho,\xi} - w)\big|_{\partial_1} = (v_{\rho,\xi} - H_{v_{\rho,\xi}})\big|_{\partial_1} = 0$$

$\Rightarrow \exists\, x^* \in \partial_2$ s.t.

$$\underbrace{v_{\rho,\xi}(x^*)}_{= v(x^*)} > w(x^*) \geq v_{R,x_0}(x^*)$$

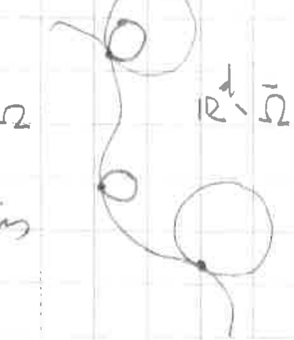
Contradiction with (a). □

2.5. <u>Dir. problem.</u>

<u>Def.</u> $\Omega \Subset \mathbb{R}^d$ satisfies the <u>exterior</u> <u>sphere</u> <u>condition</u> (ESC) if $\forall\, x^* \in \partial\Omega$ $\exists$ ball $B_R(x^*) \subset \mathbb{R}^d \setminus \bar{\Omega}$ s.t.

$$\partial B_R(x^*) \cap \partial\Omega = \{x^*\}.$$

<u>Note:</u> Ω a C^1-domain $\Rightarrow$ (ESC) holds.



<u>HW:</u> Come up with domains <u>not</u> satisfying (ESC).

<u>Thm.</u> (O Perron 1923 (Math. Z.)) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain sat. (ESC). Then for all $\varphi \in C(\partial\Omega)$ there exists a unique $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying

$$\Delta u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = \varphi.$$

Pf. Let

$$\sigma(\varphi;\Omega) := \{ v \in \sigma(\Omega) \cap C(\bar{\Omega});\ v|_{\partial\Omega} \leq \varphi \}$$

$$\Sigma(\varphi;\Omega) := \{ v \in \Sigma(\Omega) \cap C(\bar{\Omega});\ v|_{\partial\Omega} \geq \varphi \}$$

<u>Observation 1</u>

This sets are nonempty: Choose constants m and M

$$m \le \varphi \le M \quad \text{on } \partial\Omega.$$

Then constant fndns m & M are harmonic $\Rightarrow$ $m \in \sigma(\varphi;\Omega)$,

$M \in \Sigma(\varphi;M)$.

<u>Observation 2</u> Assume u is a solution; if $v \in \sigma(\varphi;\Omega)$,

Then $(v-u)|_{\partial\Omega} = 0$, $v-u$ subharmonic $\Rightarrow$

$$v - u \le 0 \text{ in } \Omega \iff v \le u.$$

Similarily, $w \in \Sigma(\varphi;\Omega) \Rightarrow u \le w$.

Hence

if a solution u exists: $v \le u \le w \quad \forall\, v \in \sigma(\varphi;\Omega), w \in \Sigma(\varphi;\Omega)$.

Define now $u_- = \sup\{v;\ v \in \sigma(\varphi;\Omega)\}$

$u_+ = \inf\{w;\ w \in \Sigma(\varphi;\Omega)\}$

Then if u exists

$$u_- \le u \le u_+.$$

The Result will follow from the next two Lemmas:

<u>Lemma a:</u> u_- (& u_+) is harmonic in Ω

<u>Lemma b:</u> u_- (& u_+) $\in C(\bar\Omega)$ and $u_\pm|_{\partial\Omega} = \varphi$.

The proofs are somewhat long & non-trivial.
Let's start by proving Lemma a):

<u>Pf. of Lemma a)</u> Fix $x_0 \in \Omega$ & choose a sequence $v_k \in \sigma(\varphi;\Omega)$ s.t. (let $u_- = u$)

$$\lim_{k\to\infty} v_k(x_0) = u(x_0).$$

(v_n) depends on x_0!

Let

$$V_n = \max\{v_1, \dots, v_n\}.$$

Then also $V_n \in \sigma(\varphi;\Omega)$ <u>and</u>

$$V_n \le V_{n+1} \text{ in } \Omega, \quad \lim_{n\to\infty} V_n(x_0) = u(x_0).$$

$\therefore$ (V_n) is an <u>increasing</u> sequence.

Assume

$$x_0 \in B_\rho(\xi) \subset \overline{B_\rho(\xi)} \subset \Omega.$$

Let $(V_n)_{\rho,\xi}$ be as defined above.

i.e. $(V_n)_{\rho,\xi}\big|_{\partial B_\rho(\xi)} = V_n$ & $(V_n)_{\rho,\xi}$ harmonic in $B_\rho(\xi)$.

Also,

$$\left[(V_n)_{\rho,\xi} - (V_{n+1})_{\rho,\xi}\right]\Big|_{\partial B_\rho(\xi)} = (V_n - V_{n+1})\Big|_{\partial B_\rho(\xi)} \leq 0$$

Max. Princ.
$\Rightarrow$

$$(V_n)_{\rho,\xi} \leq (V_{n+1})_{\rho,\xi}$$

i.e. $((V_n)_{\rho,\xi})$ is also increasing. Since

$$(V_n)_{\varphi,\xi} \in \mathcal{S}(\varphi;\Omega), \quad V_n \leq (V_n)_{\rho,\xi}$$

and $u(x_0) = \sup\{v(x_0);\ v \in \mathcal{S}(\varphi;\Omega)\}$,

$$u(x_0) = \lim_{n\to\infty} V_n(x_0),$$

We also have

$$\lim_{n\to\infty} (V_n)_{\rho,\xi}(x_0) = u(x_0).$$

Hence $\exists$

$$z(x) = \lim_{n\to\infty} (V_n)_{\rho,\xi}(x)$$

A bnd. incr.-sequence converges.

Now we have

<u>Lemma C.</u> z is harmonic in $B_\rho(\xi)$.

Let's postpone the pf. of Lemma C and see how this implies that u is harmonic.

We know

$$z(x_0) = u(x_0).$$

Fix $\hat{x} \in B_\rho(\xi)$ and let $\{\hat{v}_n\}$, $\{\hat{V}_n\}$ be sequences s.t.

$$\hat{v}_n \in \mathcal{S}(\varphi;\Omega), \quad \hat{v}_n(\hat{x}) \to u(\hat{x})$$

$$\hat{V}_n = \max\{\hat{v}_1, \dots, \hat{v}_n, V_n\}$$

We repeat the construction at $\hat{x}$!

and construct $(\hat{V}_n)_{\varphi,\xi}$ as before.

Then Lemma C) $\Rightarrow$ $\exists$ harmonic (in $B_\varphi(\xi)$) limit $\hat{z}$,

$$\hat{z}(\hat{x}) = u(\hat{x}),$$

and since

$$\hat{V}_n \geq V_n \Rightarrow (\hat{V}_n)_{\rho,\xi} \geq (V_n)_{\rho,\xi}$$

we have $\hat{z} \geq z$ in $B_\rho(\xi)$.

Hence $\tilde{z} - z \geq 0$ in $B_\rho(\xi)$. Also, def. of u as $\sup\{\dots\}$,

$$u(x_0) = z(x_0) \leq \tilde{z}(x_0) \leq u(x_0),$$

and thus $z(x_0) = \tilde{z}(x_0)$. Max. Princ. $\Rightarrow$ $\tilde{z} = z$ in $B_\rho(\xi)$,
and thus

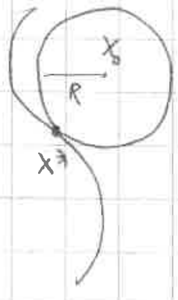
$$u(\tilde{x}) = \tilde{z}(\tilde{x}) = z(\tilde{x}) \quad \forall \tilde{x} \in B_\rho(\xi).$$

This proves Lemma a) (Mod Lemma c)).

Pf. of Lemma b): Fix $x^* \in \partial\Omega$ and let $B_R(x_0)$ be an exterior ball touching $\partial\Omega$ only at x^*

Let

$$H(x) = \begin{cases} \dfrac{1}{R^{d-2}} - \dfrac{1}{|x-x_0|^{d-2}} \\[2ex] \ln \dfrac{|x-x_0|}{R} \end{cases},$$



Then H is harmonic outside x_0 i.e. esp. in a nbhd of Ω, and

$$\begin{cases} H(x) > 0 & \text{on } \partial\Omega \setminus \{x^*\}, \\ H(x^*) = 0 \end{cases}$$

Let $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ s.t.

$$|\varphi(x) - \varphi(x^*)| < \varepsilon \quad \forall \varphi \quad x \in \partial\Omega, \ |x - x^*| < \delta.$$

We have yet another lemma:

Lemma d: $\exists$ constant $C_\varepsilon = C_\varepsilon(\|\varphi\|_\infty, R, d, \delta(\varepsilon))$ s.t.

$(\ddot\smile)$ $\forall x \in \partial\Omega$: $|\varphi(x) - \varphi(x^*)| < \varepsilon + C_\varepsilon H(x)$.

Assume this for a moment:

Now $(\ddot\smile)$ $\Rightarrow$

$$\varphi(x_*) - \varepsilon - C_\varepsilon H \leq \varphi(x) \leq \varphi(x_*) + \varepsilon + C_\varepsilon H$$

Hence

$$\varphi(x_*) + \varepsilon + C_\varepsilon H \in \Sigma(\varphi; \Omega)$$

$$\varphi(x_*) - \varepsilon - C_\varepsilon H \in \sigma(\varphi; \Omega)$$

and thus

$$\varphi(x_* - \varepsilon - C_\varepsilon H) \leq u(x) \leq \varphi(x_*) + \varepsilon + C_\varepsilon H$$

$\Leftrightarrow$

$$|u(x) - \varphi(x_*)| \leq \varepsilon + C_\varepsilon H.$$

Now when $x \to x^*$, we get $(H(x) \to 0$ as $x \to x^*)$

$$\limsup_{x \to x^*} |u(x) - \varphi(x^*)| \leq \varepsilon$$

and since $\varepsilon > 0$ was arbitrary,

$$\lim_{x \to x^*} |u(x) - \varphi(x^*)| = 0.$$

So we need supplementary lemmas c and d; c will be proven in HW - it relies on Ascoli - Arzela.

Let's prove lemma d:

Assume 1st $|x - x^*| < \delta$. Then since $H \geq 0$

$$|\varphi(x) - \varphi(x^*)| < \varepsilon < \varepsilon + C_\varepsilon H(x),$$

Let then $|x - x^*| \geq \delta$. Now $\forall\, x \in \partial\Omega$,

$$|\varphi(x) - \varphi(x^*)| \leq 2 \sup_x |\varphi(x)| \frac{H(x)}{\left(\min_{\partial\Omega \cap \{|x - x^*| \geq \delta\}} H(x)\right)}$$

when $\min_{\partial\Omega \cap \{|x - x^*| \geq \delta\}} H(x) > 0$.

Let $C_\varepsilon = \dfrac{2 \sup |\varphi(x)|}{\min_{\partial\Omega \cap \{|x - x^*| \geq \delta\}} H(x)}$. Then $|\varphi(x) - \varphi(x^*)| \leq C_\varepsilon H < \varepsilon + C_\varepsilon H$. $\square$

Function $H \geq 0$ is subharmonic in $\mathbb{R}^d \setminus \bar{\Omega}$ (although $= \infty$ at x_0)

and $\left.\begin{array}{l} H(x) = 0 \\ x \in \partial\Omega \end{array}\right\} \iff x = x^*$.

Any function like this is called a barrier function at x^*, and the $\exists$ of boundary values only need barriers at all bndry pts.

3. Heat equation

3.1. Basic questions: Heat eqn (or Diffusion eqn) with constant diffusion coefficient $k > 0$ is

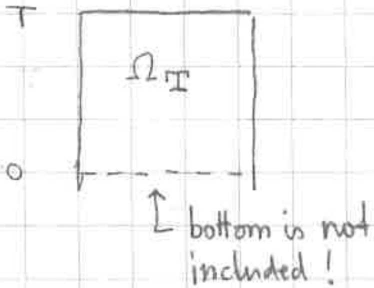
$$u_t - k\Delta u = 0. \quad t \in \mathbb{R},\ x \in \mathbb{R}^d.$$

Since we may rescale time $t' = kt$, $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial t} = k\partial_{t'} u$ we may always assume $k = 1$, and this is the equation we are going to study.

Some notations:

- If $\Omega \subset \mathbb{R}^d$, $T > 0$, let

$$\Omega_T = \Omega \times (0, T],$$



- If $\Omega = \mathbb{R}^d$, $T > 0$, let

$$S_T = \mathbb{R} \times (0, T]$$

The classical problems are the following:

Let $\mathcal{H}(\Omega_T) = \{u : \Omega_T \to \mathbb{R};\ u_t, \partial^2 u/\partial x_i x_j \in C(\Omega_T)\ \forall i,j\}$ and sim. for S_T

a) Dir. problem: Find $u \in C(\overline{\Omega_T}) \cap \mathcal{H}(\Omega_T)$ s.t.

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u|_{\partial\Omega \times [0,T]} = g \in C(\partial\Omega \times [0,T]) \\ u|_{t=0} = u_0 \in C(\overline{\Omega}) \end{cases}$$

b) Neumann-problem:

Like Dirichlet-problem, but 2^{nd} cond is replaced by $u \in C^1(\overline{\Omega}_T) \cap \mathcal{H}(\Omega_T)$,

$$\partial_n u|_{\partial\Omega \times [0,T]} = h \in C(\partial\Omega \times [0,T])$$

c) Cauchy-problem (Initial Value problem / Characteristic Cauchy problem)

Find $u \in \mathcal{H}(S_T) \cap C(\overline{S_T})$ s.t.

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } S_T \\ u|_{t=0} = f & \text{on } \mathbb{R}^d \end{cases}$$

We will mostly consider a) & c).

3.2. The fundamental Solution

Note that if $u(x,t)$ is a solution, so is $u(hx, h^2 t)$ for all h. Hence, let

$$\xi = |x|^2/t \quad \leftarrow \text{invariant under scaling } (x,t) \mapsto (hx, h^2 t),$$

so let's separate variables in ξ and t:

$$u(x,t) = h(t) f(\xi).$$

Then

$$\partial_t u(x,t) = h'(t) f(\xi) + h(t) f'(\xi) \left(-\frac{|x|^2}{t^2}\right)$$

$$\partial_{x_i} u(x,t) = h(t) f'(\xi) \frac{\partial(|x|^2/t)}{\partial x_i} = 2x_i h(t) f'(\xi)/t$$

$$\partial_{x_i x_i} u(x,t) = 2h f'/t + 4x_i^2 h(t) f''(\xi)/t^2$$

$$\Delta u(x,t) = 2d h f'/t + 4|x|^2 h f''/t^2$$

and hence

$$\partial_t u = \Delta u \iff t h'(t) f(\xi) - h(t) \xi f'(\xi)$$

$$= 2d h f' + 4 \xi h f''$$

$$\iff t h'(t) f(\xi) - 2d h f' = 4 \xi h(t) f''(\xi) + h(t) \xi f'(\xi)$$

$$= h(t) \xi \left(4 f''(\xi) + f'(\xi)\right)$$

Choose now f s.t.

$$4f'' + f' = 0 \quad \text{i.e.} \quad f'(\xi) = \hat{C} e^{-\xi/4} \Rightarrow f(\xi) = C e^{-\xi/4}.$$

Plug this into left hand side:

$$t h'(t) e^{-\xi/4} + \frac{1}{2} d h e^{-\xi/4} = 0$$

$$\iff t h' + \frac{dh}{2} = 0 \iff \underbrace{\int \frac{dh}{h}}_{\ln|h|} = -\int \frac{d\, dt}{2t} = -\ln t^{d/2}$$

i.e.

$$\begin{cases} f(\xi) = C_1 e^{-\xi/4} \\ h(t) = C_2 t^{-d/2} \end{cases}$$

and we have shown that

$$C\, t^{-d/2} e^{-|x|^2/4t}$$

is a solution $\forall\, t > 0$.

Def. $\Gamma(x,t) = \dfrac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}$ is the fundamental solution of the heat eqn. i.e. <u>the heat kernel</u>

<u>Note:</u> See Ex. 6

$$\frac{1}{[4\pi(t-s)]^{d/2}} \int e^{-|x-y|^2/4(t-s)}\, dy = 1 \quad \forall\, 0 \le s < t \quad \forall\, x \in \mathbb{R}^d \; !$$

3.3. Existence of a Solution for an initial value problem: $\Omega = \mathbb{R}^d$

<u>Thm.</u> Assume $f \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Let $\forall\, x \in \mathbb{R}^d, t>0$,

$$(\ddot{U}) \qquad u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y)\, dy .$$

Then $u \in C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, $\Delta u(x,t) = \partial_t u(x,t)$ $\forall\, t>0$ and uniformly on compact subsets of $\mathbb{R}^d$,

$$\lim_{t\to +0} u(x,t) = f(x).$$

Hence $(\ddot{U})$ is a solution of the initial value problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } t>0 \\ u|_{t=0} = f & . \end{cases}$$

<u>Pf.</u> Since all derivatives of $e^{-|x-y|^2/4t}\, t^{-d/2}$ are of the form

$$P\left(\tfrac{1}{t}, x\right) e^{-|x-y|^2/4t}, \quad P \text{ a polynomial},$$

and these are abs. integrable $\forall\, t>0$, we can take all derivatives inside integral; hence $u \in C^\infty\{t>0\}$. Also $\forall\, t>0$

$$\partial_t u - \Delta u = \int_{\mathbb{R}^d} \underbrace{(\partial_t - \Delta_x)\left\{\frac{1}{(4\pi t)^{d/2}} e^{-|x-y|^2/4t}\right\}}_{=0} f(y)\, dy = 0$$

so u solves the heat eqn. Hence it only remains to check the initial value: recall that $\forall\, t>0$

$$\frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t}\, dy = 1,$$

hence

$$u(x,t) - f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} \{f(y) - f(x)\}\, dy .$$

Let $\varepsilon > 0$ and choose $\delta > 0$ s.t.

$$|x-y| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon .$$

Now

$$\frac{1}{(4\pi t)^{d/2}} \int_{|x-y|<\delta} e^{-|x-y|^2/4t} |f(y)-f(x)|\, dy$$

$$< \varepsilon \frac{1}{(4\pi t)^{d/2}} \int_{|x-y|<\delta} e^{-|x-y|^2/4t}\, dy < \varepsilon \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t}\, dy = \varepsilon$$

and

$$\lim_{t\to +0} \frac{1}{(4\pi t)^{d/2}} \int_{|x-y|>\delta} e^{\overbrace{-|x-y|^2/4t}^{\to -\infty}} |f(y)-f(x)|\, dy \to 0$$

since $\lim_{t\to +0} t^{-N} e^{-a/4t} = 0$ $\forall\, a > \delta > 0$ + (by Monotone Convergence !)

$\varepsilon > 0$ arb. $\Rightarrow$ claim. $\square$

Uniqueness is much more subtle !

3.4. Existence of a solution for an initial value problem: Ω bnd.

Let now $\Omega \Subset \mathbb{R}^d$, and consider the problem

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u(x,t) = 0, & t>0,\ x \in \partial\Omega \\ u(x,0) = f(x), \end{cases}$$

2 deriv. in x
1 — in t

where $\overset{\text{a priori}}{f} \in C(\bar{\Omega})$ and $u \in C^{2,1}(\Omega \times \mathbb{R}_+)$

Let's separate variables:

$$u(x,t) = X(x)\,T(t).$$

Then

$$X(x)\,T'(t) = \partial_t u = \Delta_x u = \Delta X(x)\,T(t)$$

and assuming $T, X \neq 0$ we have

$$\frac{T'(t)}{T(t)} = \frac{\Delta X(x)}{X} = \text{const} = \lambda$$

Hence

$$\begin{cases} T' = \lambda T, & t > 0 \\ \Delta X = \lambda X & \text{in } \Omega,\quad X|_{\partial\Omega} = 0. \end{cases}$$

The 1st one is easy:

$$T(t) = C\,e^{-\lambda t}, \quad t > 0.$$

The 2nd is highly nontrivial for general Ω; even the existence of $X \neq 0$ sol. $\Delta X = \lambda X$ in Ω is not clear. Such X are called $\underbrace{\text{eigenvalues}}_{\text{(Dirichlet)}}$ of Δ in Ω. Let's start by taking $d = 1$ and $\Omega = [0, \pi]$.

$$\text{(EV)} \qquad \frac{d^2X}{dx^2} = \lambda X, \quad X(0) = X(\pi) = 0$$

Now for $\lambda > 0$ (EV) has only trivial sol $X = 0$:

$$\begin{cases} \dfrac{d^2X}{dx^2} = \lambda X \iff X = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x} \\ X(\pi) = C_1 e^{\sqrt{\lambda}\pi} + C_2 e^{-\sqrt{\lambda}\pi} = 0 = X(0) = C_1 + C_2 \iff C_1 = C_2 = 0 \end{cases}$$

$$\begin{vmatrix} e^{\sqrt{\lambda}\pi} & e^{-\sqrt{\lambda}\pi} \\ 1 & 1 \end{vmatrix} = e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi} \neq 0$$

Also for $\lambda = 0$:

$$\begin{cases} X'' = 0 \iff X(x) = ax + b \\ X(\pi) = a\pi + b = 0 = X(0) = b \qquad \iff a = b = 0. \end{cases}$$

Hence our only hope is $\lambda < 0$:

Then

$$X'' + \lambda X = 0 \iff X(x) = C_1 e^{i\sqrt{|\lambda|}x} + C_2 e^{-i\sqrt{|\lambda|}x}$$

$$\begin{cases} X(\pi) = C_1 e^{i\sqrt{|\lambda|}\pi} + C_2 e^{-i\sqrt{|\lambda|}\pi} = 0 \\ X(0) = C_1 + C_2 = 0 \end{cases}$$

$$\begin{vmatrix} e^{i\sqrt{|\lambda|}\pi} & e^{-i\sqrt{|\lambda|}\pi} \\ 1 & 1 \end{vmatrix} = e^{i\sqrt{|\lambda|}\pi} - e^{-i\sqrt{|\lambda|}\pi} = 2i\sin\left(\pi\sqrt{|\lambda|}\right)$$

$\neq 0$ if $\sqrt{|\lambda|}\pi \neq n\pi \Leftrightarrow |\lambda| = n^2$.

i.e. we can have nontrivial sols only if $\lambda = \lambda_n = -n^2$.

Then

$$C_1 e^{i\sqrt{|\lambda_n|}\pi} + C_2 e^{-i\sqrt{|\lambda_n|}\pi} = C_1 e^{in\pi} - C_2 e^{-in\pi} = 0 \Leftrightarrow C_1 = C_2.$$

Hence the eigensolutions are

$$\begin{cases} X_n(x) = C(e^{inx} - e^{-inx}) = C' \sin nx, \quad n > 0. \\ T_n(t) = C e^{-n^2 t} \end{cases}$$

and we can look for the general sol. u as a series

$$u(x,t) = \sum_{n>0} a_n e^{-n^2 t} \sin(nx) \text{ for some } a_n \in \mathbb{R}.$$

When does this work?

Cond. 1. The series has to converge so fast (i.e. a_n

have to go to 0 as $n \to \infty$) that one can differentiate this twice in x & once in t.

Cond. 2. Formally:

$$u(x,0) = \sum_{n>0} a_n \sin(nx) = f(x).$$

Hence we have to be able to decompose f into a series in terms of $\sin(nx)$ i.e. Fourier sine-series!

Let's worry about Cond. 2 1st. This leads into

3.5. Fourier-series: Crash course.

Let $f \in L^1([-\pi,\pi])$ and ask a general question: when can one write

$$(F) \qquad f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

for some $a_n \in \mathbb{C}$ & s.t. the series in (F) converges in some sense & "=" holds, at least a.e.?

Now formally

$$\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \sum_{n=-\infty}^{\infty} a_n \int_{-\pi}^{\pi} e^{i(n-k)x}\,dx.$$