Partial Differential Equations
University of Helsinki, Department of Mathematics and Statistics
Fall 2015
Home assignment 9

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Problem set I

1. Assume that a smooth function u solves the heat equation $u_t - \Delta u = 0$. Show that also the function
$$v(x,t) = \langle x, \nabla u(x,t)\rangle + 2tu_t(x,t)$$
solves the heat equation

2. Prove the Weierstrass approximation theorem: A continuous function on a closed interval $[a,b]$ can be uniformly approximated with a polynomial. **Hint**: Extend f to a continuous function on whole $\mathbb{R}$, and solve the heat equation with this as an initial value. Then expand the heat Kernel into a uniformly convergent power series.

3. Show that a solution of the initial value problem
$$u_t - u_{xx} = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}_+,$$
$$u(x,0) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$
is given by
$$u(x,t) = \frac{1}{2}\left(1 + E(x/\sqrt{4t})\right),$$
where the so called Error function E is defined by the formula
$$E(s) = \frac{2}{\sqrt{\pi}}\int_0^s e^{-r^2}\,dr.$$

4. Assume that u is a smooth positive function solving the heat equation
$$u_t - \mu u_{xx} = 0,$$
where the constant μ is positive. Prove that the function
$$v = -2\mu u_x/u$$
is a solution of the Burger's equation
$$v_t + vv_x = \mu v_{xx}.$$

5. Assume that the functions $u_i(s,t) : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$, $i = 1,\dots,d$, solve the one dimensional Heat equation $v_t + v_{ss} = 0$. Prove that the function
$$u(x,t) = u_1(x_1,t)\cdots u_d(x_d,t) = \Pi_{i=1}^d u_i(x_i,t)$$
is a solution of the d–dimensional Heat equation.

Problem set II

For the next three exercises read the about Ceasáro summing in the notes of the Fourier–analysis course.

1. How is defined the Cesáro sum of a 2π–periodic function?
2. Explain intuitively how this is different from summing a Fourier series using partial sums and Dirichlet kernel.
3. Formulate and prove a theorem about the Cesáro sums of a continuous 2π–periodic function.