Partial Differential Equations
University of Helsinki, Department of Mathematics and Statistics
Fall 2015
Home assignment 7

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Problem set I

The next four exercises are meant to give an idea what the first midterm exam might look like.

1. Assume that $\nabla \cdot a(x)\nabla u(x) > 0$ in the domain Ω, and that $a \in C^1(\Omega)$ is positive. Show that the function u can't have a local maxima in Ω..
2. Formulate the Harnack–principle for solutions of Laplacian, and prove it.
3. Determine all harmonic polynomials in two variables of order three or less.
4. Show that the function
$$f(x) = \frac{\sin(k|x|)}{|x|}$$
is a C^2 –solution of the Helmholtz–equation
$$(\Delta + k^2)u = 0$$
in $\mathbb{R}^3$. Do solutions of Helmholtz–equation always satisfy a maximum principle?

Problem set II

We continue proving the Ascoli–Arzéla Theorem.

5. Prove, that a closed and totally bounded subset of a metric space is compact.

In view of this, and the exercises 3 and 4 in HW 6 we have now proven, that in a compact metric space compact sets are precisely the closed and totally bounded subsets.

Let now X be a compact topological space, and $C(X)$ the vector space of all continuous functions $X \to \mathbb{C}$. Consider known, that equipped with sup–norm
$$\|f\| = \sup_{x \in X} |f(x)|,$$
$C(X)$ is a complete normed space, i.e a *Banach–space*. Let now $\Phi \subset C(X)$, which is *pointwise bounded*, i.e for every $x \in X$
$$\sup\{|f(x)]; \ f \in \Phi\} < \infty,$$
and equicontinuous, i.e for all $\varepsilon > 0$ and all $x \in X$ there is a neighbourhood V of x such that for all $f \in \Phi$ we have
$$|f(x) - f(y)| < \varepsilon \quad \text{for all } y \in V.$$

6. Prove that Φ is *uniformly bounded*, i.e there exists a constant $M \in \mathbb{R}$ such that
$$\sup\{|f(x)|; \ f \in \Phi \ x \in X\} \leq M.$$
7. Prove that Φ is totally bounded.
8. (Ascoli–Arzéla) Show that every sequence in Φ has a subsequence converging uniformly to an element of $C(X)$.