Partial Differential Equations
University of Helsinki, Department of Mathematics and Statistics
Fall 2015
Home assignment 4

Return by: ma 28.9.2015 klo 19.30
Return corrections by: ma 05.10.2015 klo 19.30

Problem set I

The purpose of the following three exercices is to explain why one cannot use Green's Theorems directly in the whole domain when proving the Representation Theorem.

1. Let
$$\phi(x) = |x|, x \in \mathbb{R}.$$
Evaluate ϕ' and ϕ''

2. Let $I = [-1, 1]$ and $f \in C_0^2(I)$. Prove by evaluating the limits of the integrals
$$\int_{I\setminus[-\varepsilon,\varepsilon]} f\phi'\,dx, \quad \int_{I\setminus[-\varepsilon,\varepsilon]} f'\phi\,dx$$
as $\varepsilon \to +0$, that
$$\int_{I\setminus\{0\}} f\phi'\,dx = -\int_I f'\phi\,dx.$$

3. Using the same ideas as in the previous exercise, show that the equation
$$\int_{I\setminus\{0\}} f\phi''\,dx = \int_I f''\phi\,dx$$
is not valid. How can you modify this so that it would hold?

Problem set II

4. Prove that *the Poisson Kernel*
$$K(x, y) = \frac{R^2 - |x|^2}{|x - y|^d}$$
is harmonic with respect to x in the open ball $B_R = \{x \in \mathbb{R}^d; |x| < R\}$ for all $y \in \partial B_R$. **Hint**: Look at the exercise 2.3.2 on page 74 in DiBenedetto.

5. Prove that formula on page 46 in DiBenedetto,
$$-\frac{\partial}{\partial|y|}G(x, y)|_{y\in\partial B_R} = \frac{1}{R\omega_N}\frac{R^2 - |x|^2}{|x - y|^N}$$
is true. Here N is the dimension of the space. **Hint**: You may also have a look in 2.2.c) in the book of Evans.

For the following two problem sets read the section 2.4 of Dibenedetto, *Subharmonic Functions and the Mean Value Property* and especially the subsection 2.4.2 *Structure of Sub-harmonic Functions*. If you need to recall the claim of Jensen's inequality, see Appendix B in Evans. Also read section 2.5.1, *The Harnack Inequality and the Liouville Theorem* in DiBenedetto.

Problem set III

6. Is the function $\max\{x_1^2 + x_2^2, x_1^2 - x_2^2\}$ subharmonic?

7. Is the function $(x_1^2 + x_2^2)^2$ subharmonic?

Problem set IV

8. Explain in your own words, what the Harnack's Theorem (Theorem 2.5.1) says.

9. Explain in your own words, what the Liouville's Theorem (Theorem 2.5.1) says.