Partial Differential Equations
University of Helsinki, Department of Mathematics and Statistics
Fall 2015
Home assignment 3

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Problem set I

1. Assume that F is a C^1–vector field and g is a differentiable real valued function. Derive the product rule of differentiation for the Divergence, i.e. compute $\nabla \cdot (gF)$.
2. Assume that u and v are C^2–functions. Expand $\Delta(uv)$.

Problem set II

3. Consider the *Helmholtz–equation*
$$(\Delta + c)\, u = 0 \quad \text{in } \mathbb{R}^3.$$
Prove that if u is radial solution of this equation, i.e. $u(x) = V(|x|)$, then the function V solves the ordinary differential equation
$$V'' + \frac{2}{r}V' + cV = 0,\ r \neq 0. \tag{1}$$
4. Solve (1). **Hint**: Look for the solution in the form $V = W/r$, and derive an equation for W.
5. Prove a representation theorem (i.e a Stokes identity using the terminology of DiBenedetto) for solutions of Helmholtz equation in bounded C^1–domains of $\mathbb{R}^3$.

For the following two problem sets read the section 2.4 of Dibenedetto, *Subharmonic Functions and the Mean Value Property* and especially the subsection 2.4.1 *The Maximum Principle*

Problem set III

6. In what plane domains the following functions are harmonic, subharmonic or superharmonic:
$$x_1^2 + x_2^2, \quad x_1^2 - x_2^2, \quad x_1x_2, \quad x_1^4x_2 + x_1x_2^4?$$
7. Compute the integral
$$\int_{B_1((1,1))} (x_1^4 - 6x_1^2x_2^2 + x_2^4)\, dS(x),$$
where $B_1((1,1))$ is the circle with radius one and centre $(1,1)$. **Hint**: Compute first $\Delta(x_1^4 - 6x_1^2x_2^2 + x_2^4)$. What do you observe?

Problem set IV

8. Explain in your own words what the maximum principle says about the behaviour of harmonic functions.
9. Consider the function
$$f(x_1, x_2) = x_1x_2$$
in a plane domain $E = \{(x_1, x_2); x_1 > 0,\ x_2 > 0\}$. Is this harmonic? Does the Maximum Principle hold? Explain what is going on.