Partial Differential Equations
University of Helsinki, Department of Mathematics and Statistics
Fall 2015
Home assignment 12

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Problem set I

1. Solve the Cauchy–problem
$$(\partial_{tt} - c^2\Delta)u = 0, \quad x \in \mathbb{R}^3,\ t > 0,$$
$$u(x,0) = |x|^2, \quad u_t(x,0) = x_3.$$

2. Let $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ be the solution of the Cauchy–problem
$$(\partial_{tt} - c^2\Delta)u = 0, \quad t > 0,$$
$$u|_{t=0} = f, \quad u_t|_{t=0} = g,$$
Assume that f and g have compact supports. Prove that there exists a constant C such that
$$|u(x,t)| \leq C/t, \quad x \in \mathbb{R}^3,\ t > 0.$$

3. Consider the same problem as in the previous exercise, but now in dimension $2+1$, i.e. $x \in \mathbb{R}^2$ and $t \geq 0$. What estimate can you now prove for $|u(x,t)|$?

4. Consider the Cauchy–problem for the *Telegraph equation*:
$$u_{tt} - u_{xx} + \lambda^2 u = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u|_{t=0} = f, \quad u_t|_{t=0} = g,$$
where $f \in C^3(\mathbb{R})$ and $g \in C^2(\mathbb{R})$. Prove, that
$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} J_0\left((\lambda\sqrt{t^2-(x-s)^2}\right) g(s)\,ds + \frac{\partial}{\partial t}\int_{x-t}^{x+t} J_0\left(\lambda\sqrt{t^2-(x-s)^2}\right) f(s)\,ds,$$
where
$$J_0(s) = \frac{2}{\pi}\int_0^{\pi/2} \cos(s\sin(\theta))\,d\theta$$
is the *Bessel function of order zero.*

Problem set II

Let H be a normed space. The *norm of a linear map* $A : H \to H$ is defined by
$$\|A\| = \sup_{x\neq 0} \|Ax\|/\|x\|.$$
This is finite if and only if A is continuous.

1. Assume that H is a complete normed space, i.e. *a Banach space*. Assume that K is a continuous linear map with $\|K\| < 1$. Prove that the equation

$$(\mathbf{I} + K)x = y, \quad y \in H,$$

has a unique solution $x \in H$ which is given by the norm convergent series

$$x = \sum_{i=0}^{\infty} (-1)^i K^i x.$$

This is the *Neumann–series* after a 19th Century German mathematician Carl Neumann.

2. Assume that B_1 and B_2 are Banach spaces, and that the linear map $A_0 : B_1 \to B_2$ has a bounded inverse A_0^{-1}. Assume that $R : B_1 \to B_2$ is a continuous linear map and that

$$\|R\| < \|A_0\|.$$

Prove that also $A_0 + R$ is invertible, and determine its inverse.

Problem set III

For this set of problems recall the definition of a compact linear map and read sections 4.4–4.6 in DiBenedetto.

1. What is the definition of an almost separable integral kernel? Is the linear map $L^2 \to L^2$ defined by an almost separable kernel always compact ?

2. Give an example of a compact linear map which is not finite dimensional.