Partial Differential Equations
University of Helsinki, Department of Mathematics and Statistics
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Home assignment 11

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Problem set I

Consider the following initial value problem for the wave equation:

$$u_{tt} - u_{xx} = 0,\ t > 0, \quad u|_{t=0} = f, \quad u_t|_{t=0} = g. \tag{1}$$

A locally integrable function u is a *weak solution* of the above problem, if for all $\phi \in C_0^2(\mathbb{R} \times \overline{\mathbb{R}}_+)$ we have

$$\int_{\mathbb{R}\times\mathbb{R}_+} u(\phi_{tt} - \phi_{xx})\, dxdt = \int_{\mathbb{R}} f(x)\phi_t(x,0) - g(x)\phi(x,0)\, dx.$$

1. Prove that every C^2–solution of (1) is also a weak solution.

2. Assume that f ja g have compact supports and are piecewise continuous. Define u using the D'Alembert formula. Prove that u is locally integrable and that it is a weak solution of (1).

3. Let $f = \chi_{[0,1]}$ and $g = 0$. Determine a weak solution of (1).

Problem set II

1. Assume that $F \in C^1(\mathbb{R} \times \overline{\mathbb{R}}_+)$. Prove that the function

   $$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t+s} F(y,s)\, dyds$$

   is a twice differentiable solution of

   $$u_{tt} - u_{xx} = F,\ t > 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = 0.$$

2. Assume that $u \in C^2(\mathbb{R} \times \overline{\mathbb{R}}_+)$ is the unique solution of (1), and that f and g have compact supports. Define the *kinetic energy of* u *at time* t by

   $$k(t) = \frac{1}{2}\int_{\mathbb{R}} u_t^2(x,t)\, dx,$$

   and the *potential energy of* u *at time* t by

   $$p(t) = \frac{1}{2}\int_{\mathbb{R}} u_x^2(x,t)\, dx.$$

   Prove that

   $$k(t) = p(t)$$

   when t is large enough.

Problem set III

1. Assume that the twice differentiable vector fields E and H in $\mathbb{R}^3$ solve the *Maxwell equations in vacuum*:

$$E_t = \frac{1}{\varepsilon_0}\nabla \times H, \quad H_t = -\frac{1}{\mu_0}\nabla \times E,$$
$$\nabla \cdot E = \nabla \cdot H = 0.$$

   Prove that the coordinate functions E_i and H_i of E and H solve the scalar wave equation

$$u_{tt} - c^2 \Delta u = 0,$$

   where $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is hence the speed of light in vacuum.

Problem set IV

For this set of problems recall the definition of compactnes, and read sections 4.1–4.3 of DiBenedetto.

1. Prove that in an infinite dimensional Hilbert–space closed and bounded subsets are not always compact.

2. What is the definition of a separable integral kernel? How does one define a compact linear operator? Is the integral operator defined by a separable integral kernel always compact?