**Partial Differential Equations**
**University of Helsinki, Department of Mathematics and Statistics**
**Fall 2015**
**Home assignment 10**

Return by: mon 16.11.2015 klo 19.30
Return corrections by: mon 23.11.2015 klo 19.30

## Problem set I

In the following two problems you prove the weak maximum principle for the heat equation.

1. Assume $\Omega \subset \mathbb{R}^d$ is a bounded domain, and $T > 0$. Let

$$\Omega_T = \{(x,t);\, x \in \Omega,\, 0 < t < T\}.$$

and

$$\partial' \Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T].$$

Assume that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies in $\Omega_T$ the inequality

$$u_t - \Delta u < 0.$$

Prove that

$$\max_{\overline{\Omega}_T} = \max_{\partial' \Omega_T} u.$$

2. (*Weak maximum principle for the heat equation*) Under the same assumptions as in the previous problem, except now we assume that in $\Omega_T$ it holds

$$u_t - \Delta u \leq 0.$$

Prove that we still have

$$\max_{\overline{\Omega}_T} = \max_{\partial' \Omega_T} u.$$

**Hint:** Apply the result of the first exercise to the function $v(x,t) = u(x,t) - kt$, where $k$ is a positive constant.

## Problem set II

1. Solve the Cauchy problem

$$u_{tt} - u_{xx} = f(x,t) \quad \text{in } \mathbb{R} \times \mathbb{R}$$
$$u|_{t=0} = u_t|_{t=0} = 0,$$

where $f(x,t) = e^{x-t}$. **Hint**: find first a function $v$ such that $v_{tt} - v_{xx} = e^{x-t}$.

2. As above, but now $f(x,t) = x^2$.

**Problem set III**

For the following set of questions read sections 5.5 - 5.7 of DiBenedetto.

1. Compare formulas (6.5) on page194, and (7.3) on page 197, and explain, what is the difference between solutions of the Cauchy problem for the wave equation in dimensions three and two.

2. What can you say about the regularity of the solutions of the wave equations in dimension three? Do you notice the difference with the one dimensional case?

3. What can you say about the behaviour of solutions of the wave equation in dimension three when $t \to \infty$ assuming that the initial data is compactly supported?

4. What can you say about the behaviour of solutions of the wave equation in dimension two when $t \to \infty$ assuming that the initial data is compactly supported?