$$\begin{cases} X(\pi) = C_1 e^{i\sqrt{|\lambda|}\pi} + C_2 e^{-i\sqrt{|\lambda|}\pi} = 0 \\ X(0) = C_1 + C_2 = 0 \end{cases}$$

$$\begin{vmatrix} e^{i\sqrt{|\lambda|}\pi} & e^{-i\sqrt{|\lambda|}\pi} \\ 1 & 1 \end{vmatrix} = e^{i\sqrt{|\lambda|}\pi} - e^{-i\sqrt{|\lambda|}\pi} = 2i\sin\left(\pi\sqrt{|\lambda|}\right)$$

$$\neq 0 \text{ if } \sqrt{|\lambda|}\pi \neq n\pi \Leftrightarrow |\lambda| = n^2.$$

i.e. we can have nontrivial sols only if $\lambda = \lambda_n = -n^2$.

Then $C_1 e^{i\sqrt{|\lambda_n|}\pi} + C_2 e^{-i\sqrt{|\lambda_n|}\pi} = C_1 e^{in\pi} + C_2 e^{-in\pi} = 0 \Leftrightarrow C_1 = -C_2$.

Hence the eigensolutions are

$$\begin{cases} X_n(x) = C(e^{inx} - e^{-inx}) = C' \sin nx, & n > 0. \\ T_n(t) = C e^{-n^2 t} \end{cases}$$

and we can look for the general sol. u as a series

$$u(x,t) = \sum_{n>0} a_n e^{-n^2 t} \sin(nx) \text{ for some } a_n \in \mathbb{R}.$$

When does this work?

<u>Cond. 1.</u> The series has to converge so fast (i.e. a_n have to go to 0 as $n \to \infty$) that one can differentiate this twice in x & once in t.

<u>Cond. 2.</u> <u>Formally</u>:

$$u(x,0) = \sum_{n>0} a_n \sin(nx) = f(x).$$

Hence we have to be able to decompose f into a series in terms of $\sin(nx)$ i.e. Fourier sine-series!.

Let's worry about Cond. 2 1st. This leads into

3.5. <u>Fourier-series: Crash course.</u>

Let $f \in L^1([-\pi,\pi])$ and ask a general question: when can one write

$$(F) \qquad f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

for some $a_n \in \mathbb{C}$ & s.t. the series in $r(F)$ converges in some sense & "=" holds, at least a.e.?

Now <u>formally</u>

$$\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \sum_{n=-\infty}^{\infty} a_n \int_{-\pi}^{\pi} e^{i(n-k)x}\,dx.$$

Since
$$\int_{-\pi}^{\pi} e^{i(n-k)x}\,dx = \begin{cases} \int_{-\pi}^{\pi} dx = 2\pi, & n = k \\ \frac{1}{i(n-k)} \Big/_{-\pi}^{\pi} e^{i(n-k)x} = 0, & n \neq k, \end{cases}$$

we get
$$\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = 2\pi a_k$$

$$\Leftrightarrow \quad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\,dx,$$

Def. If $f \in L^1[-\pi,\pi]$, we define the k^{th} Fourier cf. of f by
$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\,dx.$$

The formal series

$$(F) \qquad \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

is the Fourier-series of f. The fundamental questions are

a) When does (F) converge, and in what sense?

b) When it converges, is the limit $= f(x)$?

Neither a) or b) are not always true.

Let's see what can we say about a) first. Now $|e^{ikx}| = 1 \ \forall x$, so the decay of
$$|\hat{f}(k) e^{ikx}|$$
depends only on $\underbrace{\hat{f}(k)}_{\text{the size of}}$. Note that this completely ignores the oscillatory ("sign changing") character of e^{ikx}.

Now $f \in L^1[-\pi,\pi] \Rightarrow$
$$|\hat{f}(k)| = \left| \int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx \right| \leq \int_{-\pi}^{\pi} |f| = \|f\|_{L^1[-\pi,\pi]}$$

i.e. no decay at all. How can we get some decay? Let's assume $f \in C^1[-\pi,\pi]$ and $f(\pi) = f(-\pi)$. Then if $k \neq 0$
$$\hat{f}(k) = \int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \frac{-1}{ik} \int_{-\pi}^{\pi} f(x) \frac{d}{dx}\left(e^{-ikx}\right) dx$$
$$\underset{\text{Parts}}{\overset{\text{int. by}}{=}} \frac{1}{ik} \int_{-\pi}^{\pi} f'(x) e^{-ikx}\,dx,$$

and hence
$$|\hat{f}(k)| \leq \frac{1}{|k|} \|f'\|_{L^1[-\pi,\pi]}$$

i.e. already $\hat{f}(k) = O(|k|^{-1})$ as $|k| \to \infty$. However, this is not enough to get absolute convergence of (F)

Assume now $f \in C^2([-\pi,\pi])$, $f(\pi)=f(-\pi)$, $f'(\pi)=f'(-\pi)$.
Then int by parts once more we get

$$\hat{f}(k) = \left(\frac{1}{ik}\right)^2 \int_{-\pi}^{\pi} f''(x) e^{-ikx}$$

and hence

$$|\hat{f}(k)| \le \frac{C}{k^2}, \quad |k| \neq 0,$$

and thus (F) converges absolutely. However, this does not prove that the limit $= f(x)$.

[General obs: smoothness $\longrightarrow$ add. decay of Fourier cf's]

Let's study the symmetric partial sum

$$S_N(x) = \sum_{|k|<N} e^{ikx} \hat{f}(k).$$

Now extend f to a 2π-periodic function on $\mathbb{R}$.

$$S_N(x) = \sum_{|k|\le N} e^{ikx} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iky} f(y)\,dy\right)$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{|k|\le N} e^{ik(x-y)} f(y)\,dy$$

$$= \int_{-\pi}^{\pi} D_N(x-y) f(y)\,dy,$$

where

$$D_N(x) = \frac{1}{2\pi} \sum_{|k|\le N} e^{ikx}$$

is the N^{th} Dirichlet kernel. We could try to imitate the previous proof when we showed that the convolution by the Heat Kernel has the correct initial value. So lets compute:

$$D_N(x) = \frac{1}{2\pi}\left(e^{-iNx} + e^{-i(N-1)x} + \dots 0 + e^{ix} + \dots e^{iNx}\right)$$

$$= \frac{1}{2\pi} e^{-iNx}\big(\underbrace{1 + e^{ix} + \dots}_{\text{geom. series}} + e^{i2Nx}\big)$$

$$= \frac{1}{2\pi} e^{-iNx}\left(\frac{1-e^{i(2N+1)x}}{1-e^{ix}}\right) = \frac{1}{2\pi}\,\frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ix}}$$

$$= \frac{1}{2\pi}\,\frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-ix/2} - e^{ix/2}} = \frac{1}{2\pi}\,\frac{\sin\left((N+\frac{1}{2})x\right)}{\sin(x/2)}.$$

This is not a family of good kernels: one sees that D_N is sign-changing, and one can also prove that

$$\int_{-\pi}^{\pi} |D_N(x)|\,dx \sim \ln N$$

| See Fourier-Analysis course!

However,

$$\int_{-\pi}^{\pi} D_N(x)\,dx = \frac{1}{2\pi}\sum_{|k|\le N}\int_{-\pi}^{\pi} e^{ikx}\,dx = 1$$

and not all is lost! Let's try to estimate the difference $S_N(x) - f(x)$:

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(x-y) f(y)\,dy - f(x)$$

$$\overset{\int D_N dx = 1}{=} \int_{-\pi}^{\pi} D_N(x-y)\,(f(y) - f(x))\,dy$$

$$= \int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)(x-y)\right) h(y)\,dy,$$

when we define

$$h(y) = \frac{f(y) - f(x)}{2\pi \sin\left(\frac{x-y}{2}\right)}.$$

Assume now $f \in C^1([-\pi,\pi])$ & f is 2π-periodic.

Then

$$\sin\left(\frac{x-y}{2}\right) \Big/ \frac{x-y}{2} \longrightarrow 1 \quad \text{as } x \to y$$

$\rightsquigarrow$ $h(y)$ is unif. bounded & continuous on $[-\pi,\pi]$.

Riemann-Lebesgue Lemma: $\displaystyle\int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)x\right) h(y)\,dy \longrightarrow 0$ as $N \to \infty$!

Hence we have shown that

$f \in C^1([-\pi,\pi])$ & f 2π-periodic $\Rightarrow$

$$S_N(x) \longrightarrow f(x) \quad \text{as } N \to \infty \quad \forall\, x \in [-\pi,\pi].$$

By being more clever in the summation {& analysis} this result can be made sharper. Also, if $f \in C^2$ & period, the convergence will in fact be uniform!

However, we had $\sin(nx)$, not e^{inx}. How to modify our result?

Assume now $f \in C^1[0,\pi]$ and extend f to an odd C^1-function on $[-\pi,\pi]$ & π-periodic:

$$f(x) = f(-x) \quad \forall\, x \in [-\pi,\pi].$$

$\Rightarrow$ $f(0) = 0$!

$f(\pi)$

We have

$$f(x) = \lim_{N\to\infty} S_N(x), \text{ and}$$

$$S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx},$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$$

$$\hat{f}(-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} f(x)\,dx \overset{x \to -x}{=} -\frac{1}{2\pi}\int_{\pi}^{-\pi} e^{-inx} \underbrace{f(-x)}_{=-f(x)}\,dx$$

$$= -\hat{f}(n)$$

Hence $\hat{f}(0) = 0$, and

$$S_N(x) = \sum_{1}^{N} \hat{f}(n) e^{inx} - \hat{f}(n) e^{-inx} = 2i \sum_{1}^{N} \hat{f}(n) \frac{e^{inx} - e^{-inx}}{2i}$$

$$= 2i \sum_{1}^{N} \hat{f}(n) \sin(nx).$$

Now

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$$

$$= \frac{1}{2\pi}\left(\int_{0}^{\pi} e^{-inx} f(x) + \int_{-\pi}^{0} e^{-inx} f(x)\,dx\right)$$

$$= \frac{1}{2\pi}\left(\int_{0}^{\pi} e^{-inx} f(x)\,dx - \int_{0}^{\pi} e^{inx} f(x)\,dx\right)$$

$$= \frac{-2i}{2\pi}\int_{0}^{\pi} \sin nx\, f(x)\,dx$$

$$\therefore\ S_N(x) = 2i \sum_{1}^{N} \cdot \frac{-2i}{2\pi}\left(\int_{0}^{\pi} \sin(ny) f(y)\,dy\right) \sin(nx)$$

$$= \int_{0}^{\pi} \hat{f}_{\sin}(n) \sin(nx)\,dx$$

$$\hat{f}_{\sin}(n) = \frac{2}{\pi}\int_{0}^{\pi} \sin(nx) f(x)\,dx$$

This is the Fourier-sine series. One can proceed similarly with even extensions to obtain Fourier-Cosine-series.

<u>Back to Heat Eq:</u>

Consider the initial value problem

$$\begin{cases} (\ast)_1 & u_t - u_{xx} = 0 \quad \text{in } [0,\pi] \times \mathbb{R}_+ \\ (\ast)_2 & u(x,0) = f \\ (\ast)_3 & u(0,t) = u(\pi,t) = 0 \quad \forall\, t > 0. \end{cases}$$

We also assume that $f(0) = f(\pi) = 0$ (Compatibility Condition)

Let's look for $u(x,t)$ as a series

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx).$$

Then formally

2.3. Riippuvuusalueet

2.9
7.3

$y = x/\varepsilon$

$dx = \varepsilon\, dy$

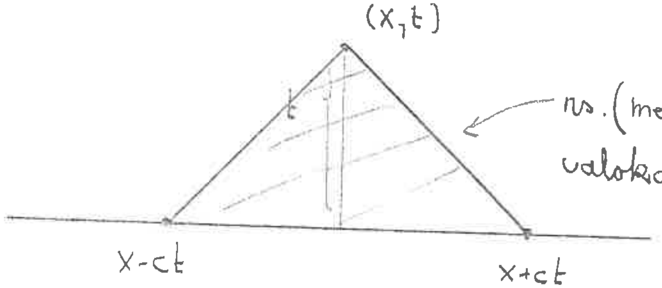
Olkoon

$$u(x,t) = \frac{1}{2}\big(\phi_0(x-ct) + \phi_0(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \phi_1(y)\,dy$$

alkuarvo-ongelman (2.1.3) yksik. ratkaisu.

Kysymys: Tarkastellaan aaltoa pisteessä x hetkellä $t>0$. Mistä osasta Cauchy-datan $u(x,t)$:n arvo riippuu?

Vast: pisteissä $x \pm ct$ ϕ_0:n arvoista
välillä $[x-ct, x+ct]$ ϕ_1:n arvoista.



ns. (menneisyyden) valokartio!

Eli pisteeseen x ajan t kuluessa informaatiota saadaan vain pisteistä $[x-ct, x+ct]$ (hajan t kuluessa).

Voimme myös kysyä seuraavaa: jos lähetämme "signaalin" pisteestä $x=0$ hetkellä $t=0$, mihin asti informaatio ehtii levitä ajan t_0 kuluessa?

2.10
7.4

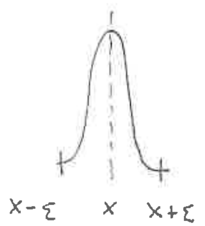
Mallinnamme tätä alkusignaalia "pulssille" ϕ_0^ε:

$$\phi_0^\varepsilon = \varepsilon^{-1/2}\phi(x/\varepsilon), \quad \phi \in C_0^2(\mathbb{R}), \quad \phi(0) = 1, \quad \phi(y) = 0$$

$$\phi_1 = 0 \qquad 0 \le \phi \le 1 \quad \text{kun } |y| \ge$$

$\varepsilon^{-1/2}$, jotta L^2-energia $\left(\int |\phi_0^\varepsilon|^2 dy\right)^{1/2}$ pysyy vakiona kun $\varepsilon \to +0$

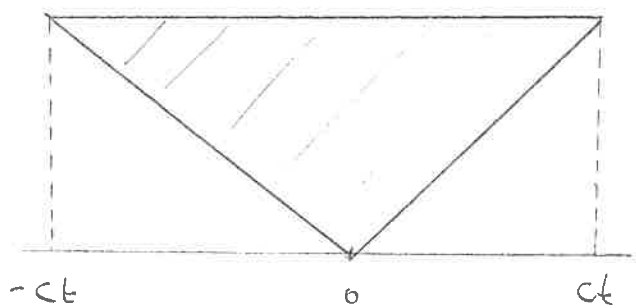
Siis $\phi_0^\varepsilon(y) = 0$ kun $|y| \ge \varepsilon$



Jos u_ε on vast. alkuarvo-ongelman ratkaisu:

$$u_\varepsilon(x,t) = \frac{1}{2}\left(\phi_0^\varepsilon(x-ct) + \phi_0^\varepsilon(x+ct)\right),$$

eli ajan t kuluessa signaali näkyy välillä $[-ct-\varepsilon, ct+\varepsilon]$; antamalla $\varepsilon \to +0$ saamme että signaali ehtii edetä matkan ct.



"signaalin äärellinen etenemis-nopeus"

tulevaisuuden valokartio

Samoin, jos 0 korvataan m.v. x:llä.

2.4. Reuna-alkuarvo-ongelma

Käytännössä ei x tai aika voi saada m.o. reaalisia arvoja: esim. värähtelevällä kielellä on äärellinen pituus. Tarkastellaan seuraavaa ongelmaa (ns. reuna-alkuarvo-ongelma): $(u \in C^2)$

$$(2.4.1)_a \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \ell, \ t > 0.$$

$$(2.4.2)_b \quad u(0,t) = u(\ell,t) = 0, \ t \geq 0$$ (päätepisteet pidetään paikallaan!)

$$(2.4.3)_c \quad u(x,0) = \phi_0(x), \quad 0 < x < \ell.$$

$$(2.4.4)_d \quad \frac{\partial u(x,0)}{\partial t} = \phi_1(x), \quad 0 < x < \ell.$$

Ratkaisu ei nyt onnistu kuten aikaisemmin; on kuitenkin helppo osoittaa, että ratkaisuja on **korkeintaan** yksi! Tämä voidaan osoittaa tarkastelemalla ns. energia-integraalia $E(t)$:

$$E(t) = \frac{1}{2}\int_0^\ell \left|\frac{\partial u(x,t)}{\partial t}\right|^2 + c^2\left|\frac{\partial u(x,t)}{\partial x}\right|^2 dx$$

"kineettinen energia" "potentiaali energia"

Pätee seuraava "energian säilymislaki":

Lause 2.4.1. $E(t)$ on t:n funktiona $(t \geq 0)$ vakio. (u (2.4.1):n ratk.)

Tod. Riittää os. että

(i) $\frac{dE(t)}{dt} = 0, \quad t > 0$

(ii) & E jva kun $t \geq 0$.

(ii) on selvä, sillä $u \in C^2([0,\ell] \times [0,\infty))$ (oletus!)

Os. (i)!

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_0^\ell 2\left(\frac{\partial u(x,t)}{\partial t}\cdot\frac{\partial^2 u(x,t)}{\partial t^2} + c^2\frac{\partial u(x,t)}{\partial x}\cdot\frac{\partial^2 u(x,t)}{\partial x \partial t}\right)dx$$

os.int. jälk. termiin

$$= \frac{1}{2}\int_0^\ell 2\left(\frac{\partial u(x,t)}{\partial t}\cdot\frac{\partial^2 u(x,t)}{\partial t^2} - c^2\frac{\partial^2 u(x,t)}{\partial x^2}\frac{\partial u(x,t)}{\partial t}\right)dx$$

$$+ \frac{1}{2}\Big/_0^\ell 2c^2\frac{\partial u(x,t)}{\partial x}\cdot\frac{\partial u(x,t)}{\partial t}\,dx$$

$$= \int_0^\ell \frac{\partial u(x,t)}{\partial t}\underbrace{\left(\frac{\partial^2 u(x,t)}{\partial t^2} - c^2\frac{\partial^2 u(x,t)}{\partial x^2}\right)}_{=0}dx$$ ← u alkuperäisen ratk.

$$+ c^2\left(\frac{\partial u(\ell,t)}{\partial x}\frac{\partial u(\ell,t)}{\partial t} - \frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t}\right)$$

Nyt

$$u(0,t) = u(\ell,t) = 0, \; t \geq 0$$

$$\Rightarrow \frac{\partial u(0,t)}{\partial t} = \frac{\partial u(\ell,t)}{\partial t} = 0, \; t \geq 0,$$

ja siis myös sijaitustermi häviää.: $\frac{dE(t)}{dt} = 0.$ □

Yksikäsitteisyys seuraa nyt helposti; riittää tark. ongelmaa (2.4.1) kun $\phi_0 = \phi_1 = 0$ ja os. että $u \equiv 0$ (Lineaarisuus!)

Lause 2.4.2. Ol. $\phi_0 = \phi_1 = 0$. Tällöin $u \equiv 0$, kun $\phi_0 \in C^1$, ϕ_1 jva

Tod. Nyt $E(t)$ vakio $\Rightarrow$

$$E(t) = E(0) = \frac{1}{2}\int_0^\ell \Big|\underbrace{\frac{\partial u(x,0)}{\partial t}}_{=\phi_1(x)=0}\Big|^2 + c^2 \Big|\underbrace{\frac{\partial u(x,0)}{\partial x}}_{=\phi_0'(x)=0}\Big|^2 dx = 0$$

Siis

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x} = 0 \quad \forall x \in [0,\ell], \; t > 0$$

eli u vakio $\Rightarrow u = 0$ sillä $u(0,t) = 0 \;\forall t > 0$. □

Olemassaolo ei seuraa yhtä helposti.

2.5. Muuttujien separointi

Haetaan ~~...~~ ratkaisua ongelmalle

$$(2.5.1)_a \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \ell, \; t > 0$$

$$(2.5.1)_b \quad u(0,t) = u(\ell,t) = 0, \quad t \geq 0.$$

Yritetään löytää u muodossa

$$u(x,t) = X(x)T(t) \qquad \text{"muuttujien separointi"}.$$

Nyt

$$\frac{\partial u(x,t)}{\partial x} = X'(x)T(t), \quad \frac{\partial^2 u(x,t)}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial u(x,t)}{\partial t} = X(x)T'(t), \quad \frac{\partial^2 u(x,t)}{\partial t^2} = X(x)T''(t)$$

Siis

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = X(x)T''(t) - c^2 X''(x)T(t),$$

eli pisteissä joissa $u(x,t) \neq 0$ ($\Rightarrow X(x), T(t) \neq 0$)

$$\frac{c^2 X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

Nyt vasen puoli ei riipu t:stä, oikea puoli ei riipu x:stä, eli molemmat ovat vakioita:

$$\frac{c^2 X''(x)}{X(x)} = \lambda = \frac{T''(t)}{T(t)}, \quad \lambda \in \mathbb{R}.$$

Eli saamme kaksi tavallista lineaarista DY:tä:

$(2.5.2)_a \; X'' - \lambda c^2 X = 0, \quad 0 < x < \ell$

$(2.5.3) \; T'' - \lambda T = 0, \quad t > 0.$

Näillä ei suinkaan ole sopivia ratkaisuja $\forall \lambda$, sillä haluamme myös että

$$u(0,t) = u(\ell,t) = 0 \quad \forall t \geq 0,$$

eli vaaditaan samantien

$$(2.5.2)_b: \quad X(0) = X(\ell) = 0.$$

Ratkaistaan nyt $(2.5.2)_a - (2.5.2)_b$:

KY on

$$r^2 - \lambda c^2 = 0 \iff r = \pm\sqrt{\lambda c^2} = \pm c\sqrt{\lambda}.$$

- Jos $\lambda = 0$, niin perusjärj. on

$$C_0, C_1 x,$$

ja yleinen ratk. on

$$X(x) = C_0 + C_1 x.$$

Nyt

$$X(0) = 0 \Rightarrow C_0 = 0,$$
$$X(\ell) = 0 \Rightarrow C_1 = 0$$

$\Rightarrow X(x) = 0$ eli vain triviaaliratkaisu.

- Jos $\lambda > 0$, saamme ratkaisut (PJ) $e^{\pm c\sqrt{\lambda}x}$,

eli

$$X(x) = C_1 e^{c\sqrt{\lambda}x} + C_2 e^{-c\sqrt{\lambda}x}.$$

Nyt

$$0 = X(0) = C_1 + C_2, \text{ eli } C_2 = -C_1$$

$$\therefore X(x) = C_1\left(e^{c\sqrt{\lambda}x} - e^{-c\sqrt{\lambda}x}\right) = 2C_1 \sinh(c\sqrt{\lambda}x)$$

$$X(\ell) = 2C_1 \sinh(c\sqrt{\lambda}\,\ell) = 0 \underset{\ell \neq 0}{\iff} C_1 = 0.$$

Siis taas vain triviaaliratkaisu $X(x) = 0$!

- Ol. lopuksi $\lambda < 0$. Nyt yleinen ratk.:

$$X(x) = C_1 \cos(c\sqrt{|\lambda|}x) + C_2 \sin(c\sqrt{|\lambda|}x)$$

$$0 = X(0) = C_1 \quad \therefore$$

$$X(x) = C_2 \sin(c\sqrt{|\lambda|}x).$$

$$0 = X(\ell) = C_2 \sin(\ell c\sqrt{|\lambda|}) = 0$$

$$\iff C_2 = 0 \text{ tai } \ell c\sqrt{|\lambda|} = m\pi, \quad m \in \mathbb{Z}$$

$$\boxed{\iff |\lambda| = \frac{m^2\pi^2}{\ell^2 c^2}}$$

Nämä me haluamme!

6.3. <u>Ratkaisu tasossa</u>

Tutkitaan nyt Cauchy-ongelmaa tasossa $x \in \mathbb{R}^2$, $t > 0$:

$$(\ast)_2 \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta_x u = 0, \; t > 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x). \end{cases}$$

Ol. että $u \in C^2(\mathbb{R}^2 \times \overline{\mathbb{R}}_+)$ on ratkaisu; määritellään $\bar{u} \in C^2(\mathbb{R}^3 \times \overline{\mathbb{R}}_+)$:aan s.e. se on vakio muuttujan x_3 suhteen:

$$\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad x_1, x_2 \in \mathbb{R}, \; t > 0.$$

Jatketaan jos

$$\bar{g}(x_1, x_2, x_3, t) = g(x_1, x_2, t)$$

$$\bar{h}(x_1, x_2, x_3, t) = h(x_1, x_2, t),$$

niin $\bar{u}$ toteuttaa Cauchy-ongelman

$$(\ast)_3 \begin{cases} \dfrac{\partial^2 \bar{u}}{\partial t^2} - c^2 \Delta_{(x, x_3)} \bar{u} = 0, \; t > 0 \\ \bar{u}(x, x_3) = \bar{g}(x, x_3) \end{cases}$$

$$\bar{u}_t(x, x_3, \,) = \bar{h}(x, x_3, t).$$

Käytetään nyt Kirchhoffin kaavaa.

Siis

$$(*)\begin{cases} u_{tt} - c^2 \Delta u = 0 &, x \in \mathbb{R}^2, t > 0 \\ u|_{t=0} = g \\ u_t|_{t=0} = h \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \bar{u}_{tt} - c^2 \Delta \bar{u} = 0 &, \bar{x} = (x, x_3) \in \mathbb{R}^3, t > 0 \\ \bar{u}|_{t=0} = \bar{g} &, t=0 \\ \bar{u}_t|_{t=0} = \bar{h} &, t=0, \end{cases}$$

kun $\bar{u}(\bar{x},t) = u(x,t)$, $\bar{g}(\bar{x}) = g(x)$, $\bar{h}(\bar{x}) = h(x)$.

Siis Kirchoff $\Rightarrow$ (Valitaan $\bar{x} = (x,0)$) Vrt $\tilde{G}$ lause!

$$\bar{u}(\bar{x},t) = \frac{t}{4\pi c^2 t^2} \int_{|\bar{y}-\bar{x}|=ct} \bar{h}(\bar{y}) + \frac{\partial}{\partial t}\left(\frac{t}{4\pi c^2 t^2} \int_{|\bar{y}-\bar{x}|=ct} \bar{g}(\bar{y})\, d\bar{s}_{\bar{y}}\right).$$

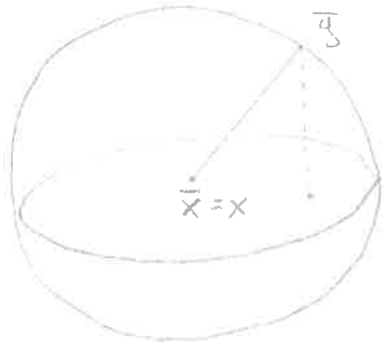
Nyt

$$\frac{1}{4\pi c^2 t^2} \int_{|\bar{y}-\bar{x}|=ct} \bar{g}(\bar{y})\, dS(\bar{y}) = \frac{1}{4\pi c^2 t^2} \int_{\substack{|\bar{y}-\bar{x}|=ct \\ y_3>0}} + \int_{\substack{|\bar{y}-\bar{x}|=ct \\ y_3<0}} \bar{g}(\bar{y})\, dS(\bar{y}).$$

Merk.

$$y_3^2 = \gamma(y) = (c^2 t^2 - |y-x|^2);$$

$$y_3 = \pm \gamma(y) = (c^2 t^2 - |y-x|^2)^{1/2}$$



Tällöin

$$dS(\bar{y}) = \pm(1 + |\nabla\gamma(y)|^2)^{1/2}\, dy\ ; \quad \begin{cases} + : \text{kun } y_3 > 0 \\ - : \text{kun } y_3 < 0 \end{cases}$$

Koska

$$\nabla\gamma(y) = \frac{y-x}{(c^2t^2 - |y-x|^2)^{1/2}},$$

niin

$$dS(y) = \left(1 + \frac{|y-x|^2}{c^2t^2 - |y-x|^2}\right)^{1/2} dy = \frac{ct}{(c^2t^2 - |y-x|^2)^{1/2}},$$

ja siis

$$\frac{t}{4\pi c^2 t^2} \int_{|\bar{y}-\bar{x}|=ct} \bar{g}(\bar{y})\, dS(\bar{y}) = \underbrace{\frac{2ct^2}{4\pi c^2 t^2}}_{=\frac{1}{2\pi c}} \int_{|y-x|<ct} \frac{g(y)}{(c^2t^2 - |y-x|^2)^{1/2}}\, dy$$

ja samoin

$$\frac{t}{4\pi c^2 t^2} \int_{|\bar{y}-\bar{x}|<ct} \bar{h}(\bar{y})\, dS(\bar{y}) = \frac{1}{2\pi c} \int_{|y-x|<ct} \frac{h(y)}{(c^2t^2 - |y-x|^2)^{1/2}}\, dy.$$

and $\exists$ limit

$\mathcal{L}(H,H) \ni B := I + A + \dots + A^n + \dots$

Let's prove that $(I-A)B = B(I-A) = I$. Let

$$B_n = I + A + \dots + A^n.$$

Then

$$(I-A)B_n = I - A^{n+1},$$

$$B_n(I-A) = I - A^{n+1},$$

so since $\|A\| < 1$, $\|A^{n+1}\| \le \|A\|^{n+1} \xrightarrow[n\to\infty]{} 0$ and thus

$$I - A^{n+1} \xrightarrow[\mathcal{L}(H,H)]{} I$$

and

$$(I-A)B = \lim_n (I-A)B_n = I = \lim_n B_n(I-A) = B(I-A)$$

This proves the Thm. $\square$

["Small perturbations of identity are still invertible"]

Before moving to applications to integral eqns, let's prove the following important consequence of Thm. 2.3.1:

Prop. 2.3.2. Let H_1, H_2 be Hilbert. Let

$$\mathrm{Inv}(H_1,H_2) = \{A \in \mathcal{L}(H_1,H_2) ;\ \exists\ B \in \mathcal{L}(H_2,H_1) \text{ s.t. } AB = I_{H_2},\ BA = I_{H_1}\}$$

The set of invertible elements

Then in the operator norm topology $\mathrm{Inv}(H_1,H_2)$ is an open subset of $\mathcal{L}(H_1,H_2)$.

Pf: Let $A \in \mathrm{Inv}(H_1,H_2)$, then for any $R \in \mathcal{L}(H_1,H_2)$

$$A + R = A(I_{H_1} + A^{-1}R),$$

and when $\|A^{-1}R\| < 1$ is $I_{H_1} + A^{-1}R$ inv., and also $A+R$ as a product of inv. ops. But

$$\|A^{-1}R\| \le \|A^{-1}\|\,\|R\| < 1$$

when

$$\|R\| < (\|A^{-1}\|)^{-1}$$

Hence $\|R\| < (\|A^{-1}\|)^{-1} \Rightarrow A+R$ is inv. $\square$

Let's now apply this to integral ops: 1st we need to estimate the operator norm of an integral operator

$$Kf(x) = \int_\Omega K(x,y)f(y)\,dy,\ x \in \Omega.$$

Here we assume that K is a meas. fund. on $\Omega \times \Omega$ s.t. functions

$$x \longmapsto K(x,y),\quad y \mapsto K(x,y)$$

are resp. ind. a.e. $\forall y$ or a.e. $\forall x$.

Prop. 2.3.3 (Schur's lemma).

If

$$\sup_x \int_\Omega |K(x,y)|\,dy,\ \sup_y \int_\Omega |K(x,y)|\,dx \le M < \infty$$

then

$$K : L^2(\Omega) \to L^2(\Omega),\ \|K\| \le M.$$

Pf. This is a straightforward application of Cauchy-Schwartz; Let $f \in C_0^\infty(\Omega)$ initially. Then

$$|K u(x)| \le \int_\Omega |K(x,y)||u(y)|\,dy$$

C-S

$$\le \underbrace{\left(\int_\Omega |K(x,y)|^{1/2 \cdot 2}dy\right)^{1/2}}_{\le M^{1/2}} \left(\int_\Omega |K(x,y)|^{1/2 \cdot 2}|u(y)|^2 dy\right)^{1/2}$$

Hence integrating over x,

$$\int_\Omega |Ku(x)|^2 dx \le M \int_{\Omega\times\Omega} |K(x,y)||u(y)|^2\,dy\,dx$$

$$\le M \underbrace{\left(\sup_y \int_\Omega |K(x,y)|dx\right)}_{\le M} \int_\Omega |u(y)|^2 dy$$

$$\le M^2 \|u\|^2_{L^2(\Omega)},$$

and thus

$$\|Ku\|_{L^2(\Omega)} \le M\|u\|_{L^2(\Omega)}. \quad \square$$

Consider now the integral eqn:

$$(\ast) \qquad f(x) + \int_\Omega K(x,y)f(y)\,dy = g(x),$$

where K is a meas. function on $\Omega\times\Omega$.

Prop. 2.3.4. If

$$\sup_x \int_\Omega |K(x,y)|dy, \ \sup_y \int_\Omega |K(x,y)|dx < 1,$$

then the integral eqn $(\ast)$ has a unique solution $f \in L^2(\Omega)$ for all $g \in L^2(\Omega)$.

Pf. Apply Sch_'s lemma to

$$Kf(x) = \int_\Omega K(x,y)f(y)\,dy$$

and the result then follows from Neumann-series.

2.4. Compact operators in Hilbert spaces

The Neumann series was based on idea that "small perturbation of the identity are invertible" and we measured the size of an op using operator norm. There are other ways to control the size of an operator.

Recall that in $\mathbb{R}^n$ ($\&\ \mathbb{C}^n$) compact sets are precisely those which are closed and bounded. This is not true in the infinitely dimensional case: let H be an inf. dimensional Hilbert space; then $\exists$ inf. seq. $(u_n)_{n=1}^\infty$ of lin. indep elements. Gram-Schmidt $\Rightarrow$ we may assume $(u_n)_1^\infty$ is orthonormal:

$$\|u_n\| = 1, \quad \langle u_n, u_m\rangle = 0,\ n \ne m.$$

Note that all

$$u_n \in B = \{x \in H;\ \|x\| \le 1\} \ \text{ which is closed \& bounded}$$

On a cpt set of a metric space (normed spaces are metric) every sequence contains a convergent subsequence.

But for $(u_n)_1^\infty$:

$$n \ne m \Rightarrow \|u_n - u_m\|^2 = \langle u_n - u_m, u_n - u_m\rangle \overset{\langle u_n, u_m\rangle = 0}{=} \|u_n\|^2 + \|u_m\|^2 = 2$$

$$\Rightarrow \|u_n - u_m\| = \sqrt{2} \quad \therefore \quad (u_n)_1^\infty \text{ is not Cauchy} \Rightarrow \text{does not converge!}$$

So again

$$\{\text{cpt sets}\} \subsetneq \{\text{bnd. \& closed}\}.$$

Def. 2.4.1 A bnd op $K : H_1 \to H_2$ (Hilbert) is <u>compact</u> if it maps every bnd set U in H_1 to a precompact set in H_2 i.e. if $\overline{K(U)}$ is compact $\forall\, U \subset H_1$ bounded.
$\mathcal{K}(H_1,H_2) = \{K \in \mathcal{B}(H_1,H_2);\ K \text{ cpt}\}$

Rem. i) All linear maps to finite dimensional spaces are compact.

ii) Let $B_{H_1} = \{x \in H_1 ;\ \|x\| = 1\}$ be the closed unit ball of H_1. Then it is enough to assume that $\overline{K(B_{H_1})}$ is compact.

We'll see in HW5 the following:

a) If $A, B : H_1 \to H_2$ are cpt then $A+B$ is cpt

b) If $K : H_1 \to H_2$ is cpt then

AK is cpt $\forall\, A \in \mathcal{B}(H_2, H)$

KB —"— $B \in \mathcal{B}(H, H_1)$

are cpt

c) If $K_n \in \mathcal{K}(H_1,H_2)$ and $K \in \mathcal{B}(H_1,H_2)$, $\|K_n - K\| \xrightarrow[n\to\infty]{} 0$ then $K \in \mathcal{K}(H_1,H_2)$.

Now we'll prove the following:

Prop. 2.4.2 If $K \in \mathcal{B}(H_1,H_2)$ is cpt, then $K^* \in \mathcal{B}(H_2,H_1)$ is also cpt.

Remark

Recall that the adjoint A^* of $A \in \mathcal{B}(H_1,H_2)$ is def by

$$\langle Ax, y\rangle_{H_2} = \langle x, A^*y\rangle_{H_1} \quad \forall\, x \in H_1, y \in H_2.$$

It is an easy consequence of Riesz-rep thm that A^* exists & is bounded:

Consider $\lambda_y : H_1 \to \mathbb{C}$, $x \mapsto \langle Ax, y\rangle$.
Then λ_y is a linear functional of $H_1 \Rightarrow \exists\, z \in H_1$ s.t.

$$\underbrace{\lambda_y(x)}_{\langle Ax,y\rangle} = \langle x, z\rangle \quad ; \quad \|z\|_{H_1} = \|\lambda_y\|$$

Def. $A^*y = z$. Since

$$|\langle Ax, y\rangle| \le \|A\| \|x\| \|y\| \quad \forall\, x \in H_1$$

$\Rightarrow \|\lambda_y\| \le \|A\| \|y\|$ and thus

$$\|A^*y\|_{H_2} \le \|A\| \|y\| \quad \text{and } A^* \text{ is bnd.}$$

Linearity follows from

$$\langle x, A^*(\alpha y_1 + \beta y_2)\rangle = \bar\alpha \langle Ax, y_1\rangle + \bar\beta \langle Ax, y_2\rangle$$
$$= \langle x, \alpha A^*y_1 + \beta A^*y_2\rangle.$$

So now let's prove that K^* is cpt if K is cpt.

Pf. We need the following fundamental result of functional analysis: if $(u_n)_{n=1}^{\infty}$ is a bnd sequence of a Hilbert space H, it contains a <u>weakly convergent</u>

subsequence $(u_{n_k})_{k=1}^{\infty}$ i.e. sequence

$$(\langle u_{n_k}, y\rangle)_{k=1}^{\infty}$$

is convergent in $\mathbb{C}$ $\forall\ y \in H$ [Pf: True even for Banach spaces & bnd linear func. on them, see lectures on Funct. Analysis by Astala - TyKK]

Assume now that $(y_n)_{n=1}^{\infty}$ is a bnd sequence in H_2; we want to prove that $(K^* y_n)_1^{\infty}$ contains a convergent subsequence. Let

$$y_{n_k} \to y \text{ weakly in } H_2.$$

Then

(☆) $$\|K^* y_{n_k} - K^* y\|^2 = \langle K^* y_{n_k} - K^* y, K^* y_{n_k} - K^* y\rangle$$

$$= \langle KK^* y_{n_k} - KK^* y, y_{n_k} - y\rangle.$$

Now KK^* cpt, (y_n) bnd, so passing to a subsequence of (y_{n_k}) if necessary we may assume that

$$\|KK^*(y_{n_k} - y)\|_{H_2} \to 0,$$

but since (y_{n_k}) bnd we have by (☆)

$$\|K^* y_{n_k} - K^* y\|_{H_1}^2 \le C\|KK^*(y_{n_k} - y)\|_{H_2} \xrightarrow{k\to\infty} 0. \quad \square$$

2.5 Integral ops with weakly singular kernels

Let's now study the compactness properties of

$$Ku(x) = \int_{\Omega} k(x,y)u(y)dy, \quad \Omega \Subset \mathbb{R}^n, \quad y, x \in \Omega.$$

We will use the following result:

Thm 2.5.1 Let $\mathcal{F}(H_1,H_2) = \{F \in \mathcal{L}(H_1,H_2);\ F$ has fin. dimensional ~~image~~-space$\}$. Then (H_1, H_2 Hilbert-sp.)

$$\overline{\mathcal{F}}(H_1,H_2) = \mathcal{K}(H_1,H_2)$$

where the closure is w.r.t. operator-norm.

For the pf, see [Yosida; Functional Analysis].

Also

$$\overline{\mathcal{F}} \subset \mathcal{K}$$

is true also in Banach ~~Hilbert~~-spaces, but the inclusion

$$\mathcal{K} \subset \overline{\mathcal{F}}$$

is not always. (Also $\overline{\mathcal{F}} \subset \mathcal{K}$ is easy to prove; see HW3)

Now we can prove:

Prop 2.5.2 If Ω bnd and $K \in C(\overline{\Omega} \times \overline{\Omega})$ then $K \in \mathcal{K}(L^2(\Omega), L^2(\Omega))$.

Pf. To keep notations simple, we consider only the case when $n=1$, $\Omega = (a,b) \subset \mathbb{R}$. Given $n \in \mathbb{N}$ let

Let

$$a_k = a + \frac{k(b-a)}{n}, \quad k = 0, \dots, n$$

$$\Omega_{k,l} = [a_k, a_{k+1}] \times [b_l, b_{l+1}], \quad k,l = 0, \dots, n-1$$

and choose $x_{k,l} \in \Omega_{k,l}$.

Then K unif. cont. in $\overline{\Omega} \times \overline{\Omega}$ $\Rightarrow$

$$\sup_x \int_\Omega |(K-K^{(n)})(x,y)|\,dy, \quad \sup_y \int_\Omega |(K-K^{(n)})(x,y)|\,dx \to 0$$

where

$$K^{(n)}(x,y) = \sum_{k,l=0}^{n-1} K(x_{k,l})\, \chi_{\Omega_{k,l}}(x,y).$$

Hence if

$$\mathcal{K}^{(n)} f(x) = \int_\Omega K^{(n)}(x,y) f(y)\,dy,$$

Schur $\Rightarrow$ $\mathcal{K}^{(n)} \to \mathcal{K}$ in $\mathcal{L}(L^2(\Omega), L^2(\Omega))$. But

$$\mathcal{K}^{(n)} f(x) = \sum_{k,l=0}^{n-1} K(x_{k,l}) \int_\Omega \chi_{\Omega_{k,l}}(x,y) f(y)\,dy$$

$$= \sum_{k,l=0}^{n-1} K(x_{k,l})\, \chi_{[a_k, a_{k+1}]}(x) \int_{a_l}^{a_{l+1}} f(y)\,dy$$

$\Rightarrow$ $\mathcal{K}^{(n)} f \in \operatorname{span}\{\chi_{[a_0,a_1]}, \dots, \chi_{[a_{n-1},a_n]}\}$

hence $\mathcal{K}^{(n)}$ cpt & $\mathcal{K}$ also.

The case of $\dim \Omega = n$ follows similarly approximating Ω by a linear combination of characteristic functions of cubes, we leave this to you ☺ $\square$

Consider now a singular kernel of type

(WS) $$K(x,y) = \frac{Q(x,y)}{|x-y|^\alpha}, \quad \alpha < n, \quad Q \in C(\overline{\Omega} \times \Omega)$$

and $\Omega \Subset \mathbb{R}^n$ bdd; a kernel of this type is called <u>weakly singular</u>. If $\alpha = n$, K is an example of so called <u>Calderon–Zygmund kernel</u>, if ~~we extend~~ we allow for $\Omega = \mathbb{R}^n$ and if $\alpha > n$, then the kernel is <u>hypersingular</u>.

We will consider only weakly singular kernels.

<u>Prop. 2.5.3</u> If $\Omega \Subset \mathbb{R}^n$ bdd and K is weakly singular, then $\mathcal{K} : L^2(\Omega) \to L^2(\Omega)$ is compact.

Pf. Given $\varepsilon > 0$, let

$$K_\varepsilon(x,y) = \begin{cases} K(x,y), & |x-y| > \varepsilon, \\ \dfrac{Q(x,y)}{\varepsilon^\alpha}, & \text{if } |x-y| \le \varepsilon \end{cases}$$

Then K_ε is cont and hence

$$\mathcal{K}_\varepsilon f(x) = \int_\Omega K_\varepsilon(x,y) f(y)\,dy$$

defines a cpt op $L^2(\Omega) \to L^2(\Omega)$. Let

$$D_\varepsilon(x,y) := K(x,y) - K_\varepsilon(x,y) = \begin{cases} 0, & |x-y| > \varepsilon \\ Q(x,y)\left[\dfrac{1}{|x-y|^\alpha} - \dfrac{1}{\varepsilon^\alpha}\right], & |x-y| \le \varepsilon \end{cases}$$

Now

$$\int_\Omega |K(x,y)|\,dy \le \left(\sup_{\Omega\times\Omega} Q\right) \int_\Omega |x-y|^{-\alpha}\,dy \le C_{\Omega,\alpha} \sup_{\overline{\Omega}\times\overline{\Omega}} Q$$

int. when $\alpha < n$.

$$\int_\Omega |K(x,y)|\,dx \le \left(\sup_{\Omega\times\Omega} Q\right) \int_\Omega |x-y|^{-\alpha}\,dx \le C_{\Omega,\alpha} \sup_{\overline{\Omega}\times\overline{\Omega}} Q$$

So $K : L^2(\Omega) \to L^2(\Omega)$ is bnd. Now $\forall x \in \Omega$:

$$\int_\Omega |D_\varepsilon(x,y)|\,dx \leq \|Q\|_\infty \int_{\substack{|x-y|<\varepsilon \\ y\in\Omega}} \left|\frac{1}{|x-y|^\alpha} - \frac{1}{\varepsilon^\alpha}\right| dy$$

$$\leq \|Q\|_\infty \left(\int_0^\varepsilon r^{n-1-\alpha}\,dr + C\varepsilon^{n-\alpha} \right) \qquad \text{vol}\,\{|x-y| \leq \varepsilon\} \sim \varepsilon^n$$

$$= \|Q\|_\infty \left(\Big/_0^\varepsilon r^{n-\alpha} + C\varepsilon^{n-\alpha} \right) = O(\varepsilon^{n-\alpha}) \xrightarrow[\varepsilon\to 0]{} 0$$

Similarly

$\forall y$: $\quad \int_\Omega |D_\varepsilon(x,y)|\,dy \leq C\varepsilon^{n-\alpha}$

Hence $D_\varepsilon = K - K_\varepsilon : L^2(\Omega) \to L^2(\Omega)$ with norm $\sim C\varepsilon^{n-\alpha}$ and

hence K is cpt as a norm limit of cpt ops. □

III FREDHOLM-YHTÄLÖT JA KOMPAKTIEN OPERAATTORIEN SPEKTRI ← OPS: again in Finnish.

3.1. Definitions

Let $A \in \mathcal{B}(H_1, H_2)$; H_1, H_2 Hilbert. We define

$\ker A = \{x \in H_1 ; Ax = 0\}$ "kernel or null-space of A"

$\operatorname{im} A = \{y \in H_2 ; \exists x \in H_1 \text{ s.t. } Ax = y\}$

"image or range of A"

Note: A cont $\Rightarrow$ $\ker A$ is always a closed subspace.

$\operatorname{im} A$ might not be: Let $A : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$,

$A : (a_n)_1^\infty \mapsto (e^{-n} a_n)_1^\infty$

Then $\|A\| = 1/e$ and assume $\tilde{H} := \operatorname{im} A$ is closed.

If $x = (x_n)_1^\infty \in \tilde{H}^\perp$, then

$$0 = \langle (x_n), (e^{-n} a_n) \rangle = \sum x_n \bar{a}_n e^{-n} \quad \forall (a_n) \in \ell^2(\mathbb{C})$$

Hence (choose $a_n = \delta_{k,n}$) $x_k e^{-k} = 0 \ \forall k \Leftrightarrow x_k = 0$

$\therefore \tilde{H}^\perp = \{0\} \Leftrightarrow \tilde{H} = \ell^2(\mathbb{C})$. This cannot
↑ we assume $\tilde{H}$ closed

be true: let $b_n = 1/n$; then $(b_n) \in \ell^2(\mathbb{C})$,

and $A(a_n)_1^\infty = (b_n)_1^\infty \Leftrightarrow a_n = e^n b_n = e^n/n$.

But $(a_n)_1^\infty \notin \ell^2(\mathbb{C})$.

So

FACTS OF LIFE I: [image of a linear op is not generally a closed subspace]

Under some assumptions this will hold. To this end we define

Def 3.1.1 i) Let $K \in \mathcal{K}(H_1, H_2)$. Then an eqn

$$Kx = y, \quad y \in H_2,$$

is a Fredholm equation of 1st kind

ii) if $H = H_1 = H_2$ and $K \in \mathcal{K}(H_1, H_2)$, an equation

$$(\mathbb{I}_H - K)x = y, \quad y \in H_2,$$

is a Fredholm equation of 2nd kind.

Like with Volterra-eqns, Fredholm eqns of 2nd kind are easier to analyze than 1st kind. This we will now do.

3.2. Fredholm's alternative

In this section, $K : H \to H$ is a fixed cpt, H Hilbert

Lemma 3.2.1 $\ker(\mathbb{I} - K)$ is finite dimensional.

Pf. Assume $\dim \ker(\mathbb{I} - K) = +\infty$. Then $\exists$ infinite ON-set $u_k \in \ker(\mathbb{I} - K)$, $k = 1, 2, \ldots$ i.e.
↑ use Gram-Schmidt

III FREDHOLM-YHTÄLÖT JA KOMPAKTIEN OPERAATTORIEN SPEKTRI

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If $x = (x_n)_1^\infty \in \widetilde{H}^\perp$, then

$$0 = \langle (x_n), (e^{-n} a_n) \rangle = \sum x_n \bar{a}_n e^{-n} \quad \forall (a_n) \in \ell^2(\mathbb{C})$$

Hence (choose $a_n = \delta_{k,n}$) $x_k e^{-k} = 0 \ \forall k \Leftrightarrow x_k = 0$

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be true: let $b_n = 1/n$; then $(b_n) \in \ell^2(\mathbb{C})$,

and

$$A(a_n)_1^\infty = (b_n)_1^\infty \Leftrightarrow a_n = e^n b_n = e^n/n.$$

But $(a_n)_1^\infty \notin \ell^2(\mathbb{C})$.

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Under some assumptions this will hold. To this end we define

Def 3.1.2 i) Let $K \in \mathcal{K}(H_1, H_2)$. Then an eqn

$$Kx = y, \quad y \in H_2,$$

is a Fredholm equation of 1st kind

ii) if $H = H_1 = H_2$ and $K \in \mathcal{K}(H_1, H_2)$, an equation

$$(\mathbb{I}_H - K)x = y, \quad y \in H_2,$$

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↑ use Gram-Schmidt

$\|u_k\| = 1$, $\langle u_k, u_\ell \rangle = 0$ when $k \neq \ell$.

Now (we've seen this before) if $k \neq \ell$,

$$\|u_k - u_\ell\|^2 = \langle u_k - u_\ell, u_k - u_\ell \rangle = \underbrace{\|u_k\|^2}_{=1} + \underbrace{\|u_\ell\|^2}_{=1} - \overbrace{\langle u_k, u_\ell \rangle}^{=0} - \overbrace{\langle u_\ell, u_k \rangle}^{=0}$$

$= 2$

So $(u_k)_1^\infty$ a bd seq.; Also

$(\mathbb{I} - K)u_k = 0 \iff Ku_k = u_k$. Hence

$\|Ku_k - Ku_\ell\|^2 = \|u_k - u_\ell\|^2$

and hence $\nexists$ a convergent subsequence of $(Ku_n)_1^\infty$. ↯. □

<u>Lemma 3.2.2.</u> $\operatorname{im}(\mathbb{I} - K)$ is closed.

Pf. Let's first prove that $\exists\ \gamma \geq 0$ s.t. (const)

(1) $\forall u \in \ker(\mathbb{I} - K)^\perp$: $\|u - Ku\| \geq \gamma \|u\|$.

Assume (1) does not hold for any $\gamma > 0$. Then $\forall k\ \exists$ $u_k \in H$, $\|u_k\| = 1$,

($1/k$) $\quad \|u_k - Ku_k\| < 1/k$, $\quad u_k \in \ker(\mathbb{I} - K)^\perp$

Hence $u_k - Ku_k \xrightarrow{H} 0$. Since (u_k) bdd, $\exists$ weakly conv. subsequence (u_{k_j}). Let $u = w\text{-}\lim u_{k_j}$.

But then [illegible] moving to [illegible] subseq. once more)

K cpt $\Rightarrow$ $Ku_{k_j} \xrightarrow{H} Ku \Rightarrow u_{k_j} \to Ku$ ($1/k$)

so ($1/k$) again gives $Ku = u$. Hence $u \in \ker(\mathbb{I} - K)$

and thus

$$0 = \langle u_{k_j}, u \rangle = 0 \quad , j = 1, 2, \ldots ,$$

But $u_{k_j} \xrightarrow{w} u \Rightarrow$

$$0 = \lim_j \langle u_{k_j}, u \rangle = \|u\|^2,$$

but since $\|u_{k_j}\| \to 1$, $u_{k_j} \xrightarrow{H} u \Rightarrow \|u\| = 1$ ↯

Hence (1) holds.

Assume $\|v_k - v\| \to 0$, $v_k = u_k - Ku_k$;

and we can take $u_k \in \ker(\mathbb{I} - K)^\perp$. Then

$$\|v_k - v_\ell\| = \|\underbrace{(u_k - u_\ell)}_{\in \ker(\mathbb{I} - K)^\perp} - K(u_k - u_\ell)\| \geq \gamma \|u_k - u_\ell\|$$

$\Rightarrow (u_k)$ is a Cauchy seq. in $H \Rightarrow \exists\ u = \lim u_k$. But Then

$$(\mathbb{I} - K)u = \lim (u_k - Ku_k) = \lim v_k = v$$

$\Rightarrow v \in \operatorname{im}(\mathbb{I} - K)$. □

Before the next result we need the following general fact:

<u>Prop. 3.2.3.</u> If $A \in \mathcal{L}(H_1, H_2)$, H_1, H_2 Hilbert then

$$\overline{\operatorname{im}(A)} = \ker(A^*)^\perp$$

Pf. It is enough to show that

$$\langle Ax, y\rangle = 0 \quad \forall x \in H_1,\ y \in \ker(A^*).$$

But $A^* y = 0 \Rightarrow$

$$0 = \langle x, A^* y\rangle = \langle Ax, y\rangle \quad \forall x \in H_1. \ \square$$

<u>Prop. 3.2.4</u> If $K \in \mathcal{K}(H, H)$, then

$$\mathrm{im}(\mathbb{I} - K) = \ker(\mathbb{I} - K^*)^{\perp}.$$

Pf. Since $\mathrm{im}(\mathbb{I}-K)$ is closed, this follows from the prev. prop. $\square$

This is often said as "$\mathrm{im}(\mathbb{I}-K)$ has finite codimension". Namely, since $\mathrm{im}(\mathbb{I}-K)$ is closed we can write

$$H = \underbrace{\ker(\mathbb{I}-K^*)}_{\text{finite dimensional}} \overset{\perp}{\oplus} \ker(\mathbb{I}-K^*)^{\perp} = \ker(\mathbb{I}-K^*) \overset{\perp}{\oplus} \mathrm{im}(\mathbb{I}-K)$$

<u>Prop. 3.2.5</u> If $K \in \mathcal{K}(H, H)$, then

$$\ker(\mathbb{I}-K) = \{0\} \Leftrightarrow \mathrm{im}(\mathbb{I}-K) = H.$$

i.e. $\boxed{\mathbb{I} - K \text{ is inj} \Leftrightarrow \mathbb{I} - K \text{ is surj.}}$

Pf. Assume $\ker(\mathbb{I}-K) = \{0\}$, but

$$\mathrm{im}(\mathbb{I}-K) =: H_1 \subsetneq H$$

is a proper closed subspace of H. Let now $H_2 := (\mathbb{I}-K)H_1$.

This is again a closed subspace of H_1 & H. Assume $H_2 = H_1$.

Then $\forall x_1 \in H_1\ \exists x_2 \in H_2$ s.t.

$$(\mathbb{I}-K)x = x_1 = (\mathbb{I}-K)y_1 \quad \text{for a unique } y_1 \in H$$

$\mathbb{I}-K$ inj $(\mathbb{I}-K)$ inj

$\Rightarrow (\mathbb{I}-K)x = y_1$. Hence $H_1 = H$ ↯.

Thus

$$H \supsetneq H_1 \supsetneq H_2.$$

Generally letting $H_k = (\mathbb{I}-K)H_{k-1}$, $k = 2, 3, \dots$,

we have a strictly decreasing sequence of closed subspaces

$$H \supsetneq H_1 \supsetneq H_2 \supsetneq \dots \supsetneq H_k \supsetneq H_{k+1} \supsetneq \dots$$

Choose $u_k \in H_k$, $\|u_k\| = 1$ and $u_k \perp H_{k+1}$ ← We can do this since $\forall k$ $H_{k+1} \subsetneq H_k \Rightarrow H_k \neq \{0\}$

Then

$$(*)\ Ku_k - Ku_l = -\underbrace{(u_k - Ku_k) + (u_l - Ku_l)}_{\in H_{k+1} \subset H_{l+1}} + \underbrace{(u_k - u_l)}_{\in H_{l+1}}$$

Assume $k > l$. Then $u_k \in H_k \subset H_{l+1}$. Hence

$$-(u_k - Ku_k) + (u_l - Ku_l) + u_k \subset H_{l+1}$$

$$u_l \in H_{l+1}^{\perp},\ \|u_l\| = 1$$

So $(*) \Rightarrow$

$\|Ku_k - Ku_l\|^2 \geq \|u_l\|^2 = 1$ ∴ This is a contradiction since K cpt. Thus

(im) $\quad \mathrm{im}(\mathbb{I}-K) = H.$

Conversely assume (im) holds. Prop. 3.2.4 $\Rightarrow$

$$\ker(\mathbb{I}-K^*) = \{0\}$$

So $\quad \mathrm{im}(\mathbb{I}-K^*) = H$

$$\ker(\mathbb{I}-K)^{\perp} \Rightarrow \ker(\mathbb{I}-K) = \{0\}. \ \square$$

Prop. 3.2.6 $\dim \ker(\mathbb{I}-K) = \dim\ker(\mathbb{I}-K^*)$

Pf. It is enough to prove

$(\hat{\ast})$ $\dim\ker(\mathbb{I}-K) \geq \dim \operatorname{im}(\mathbb{I}-K)^{\perp}$,

for $\operatorname{im}(\mathbb{I}-K)^{\perp} = \ker(\mathbb{I}-K^*)$,

so $(\hat{\ast})$ implies

$(\hat{\ast})$ $\dim\ker(\mathbb{I}-K) \geq \dim\ker(\mathbb{I}-K^*)$

and the claim follows from this by applying $(\hat{\ast})$ to K^*.

To prove $(\hat{\ast})$ we argue by contradiction: assume

$(\check{\ast})$ $\dim\ker(\mathbb{I}-K) < \dim\operatorname{im}(\mathbb{I}-K)^{\perp}$.

Let

$$A : \ker(\mathbb{I}-K) \to \operatorname{im}(\mathbb{I}-K)^{\perp}$$

be bnd, linear, injective and not onto: here we use $(\check{\ast})$. Extend

$A : H \to \operatorname{im}(\mathbb{I}-K)^{\perp}$ s.t. $A|_{\ker(\mathbb{I}-K)^{\perp}} = 0$.

A is finite dimensional $\Rightarrow$ $K+A$ is cpt.

Also

$$0 = (\mathbb{I}-(K+A))u \iff u = Ku + Au \iff \underbrace{u-Ku}_{\in \operatorname{im}(\mathbb{I}-K)} = \underbrace{Au}_{\in \operatorname{im}(\mathbb{I}-K)^{\perp}}$$

$\Rightarrow$ $0 = u - Ku = Au$ $\Rightarrow \begin{cases} u \in \ker(\mathbb{I}-K) \\ Au = 0 \end{cases}$ $\underset{\text{on } \ker(\mathbb{I}-K)}{\overset{A \text{ inj}}{\Rightarrow}}$ $u = 0$.

Hence $\mathbb{I}-(K+A)$ is inj. Then Prop 3.2.5 $\Rightarrow$

$$\operatorname{im}(\mathbb{I}-(K+A)) = H.$$

This is not true: choose $v \in \operatorname{im}(\mathbb{I}-K)^{\perp}$, $v \notin \operatorname{im}(A)$.

Then if

$\underbrace{u-Ku}_{\in\operatorname{im}(\mathbb{I}-K)} - Au = v \in \operatorname{im}(\mathbb{I}-K)^{\perp}$ $\underset{\text{Thm}}{\overset{\text{Proj.}}{\Rightarrow}}$ $-Au = v \notin \operatorname{im}(A)$ $\lightning$. $\square$

3.2.7 Fredholm's alternative

We can summarize the above as follows: Let $K \in \mathcal{K}(H,H)$;

(A) if $\mathbb{I}-K$ is inj, then it is onto, and conversely if $\mathbb{I}-K$ is onto, then $\mathbb{I}-K$ is inj.

(B) if $\mathbb{I}-K$ is not inj, let $k<\infty$ be $k = \dim\ker(\mathbb{I}-K)$; Then $k = \dim\ker(\mathbb{I}-K^*) = \dim\operatorname{im}(\mathbb{I}-K)^{\perp}$ and hence eqn

$$(\mathbb{I}-K)u = v$$

has a solution u if and only if $v \perp \operatorname{im}(\mathbb{I}-K)^{\perp}$ i.e. $\exists$ ON vectors $\beta_1, \dots, \beta_k \in \operatorname{im}(\mathbb{I}-K)^{\perp}$ s.t.

$$\langle v, \beta_\ell\rangle = 0, \quad \ell = 1, \dots, k.$$

Also u is unique only modulo a subspace of dimension k.

3.3. Spectrum of cpt operators

Recall that if A is a $n\times n$ matrix, $\lambda \in \mathbb{C}$ is an eigenvalue if $A - \lambda\mathbb{I}$ is not invertible. For a general $n\times n$-matrix, eigenvalues are the (complex) roots of the polynomial (of degree n)

$$0 = \det(A - \lambda\mathbb{I}),$$

hence there are precis. n eigenvalues (counting multiplicities)