Diffusion instability (hopefully error-free now).

(Turing instability) (for numerical example: see old lecture notes)

Instability caused by the introduction of diffusion in a system that without diffusion is stable.

Example

Non-spatial model:

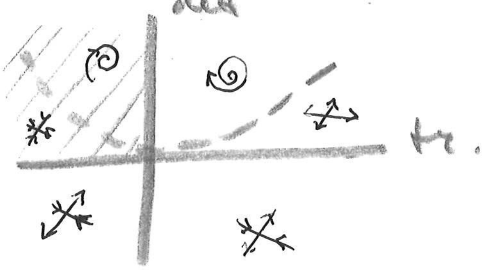
$$\begin{cases} \dot{m} = f(m,n) \\ \dot{n} = g(m,n) \end{cases}$$

Suppose (m^*, n^*) is equil:

$$\begin{cases} 0 = f(m^*, n^*) \\ 0 = g(m^*, n^*) \end{cases}$$

Assume stability:

$$\begin{cases} \operatorname{tr} A < 0 \\ \det A > 0 \end{cases}$$



where A is Jacobi-matrix.

$$A = \begin{pmatrix} \partial_m f & \partial_n f \\ \partial_m g & \partial_n g \end{pmatrix}_{\substack{m=m^* \\ n=n^*}} =: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Now add diffusion:

$$① \begin{cases} \partial_t m = f(m,n) + \mu \, \partial_{xx} m \\ \partial_t n = g(m,n) + \nu \, \partial_{xx} n \end{cases}$$

with reflecting boundaries

$\partial_x m = \partial_x n = 0$ at $x = 0, L$.

Note that (m^*, n^*) is also a (spatially uniform) equilibrium of ①.

Linearise ① about (m^*, n^*):

$$② \quad \partial_t \begin{pmatrix} \Delta m \\ \Delta n \end{pmatrix} = A \begin{pmatrix} \Delta m \\ \Delta n \end{pmatrix} + D \, \partial_{xx} \begin{pmatrix} \Delta m \\ \Delta n \end{pmatrix}.$$

where $\Delta m = m - m^*$, $\Delta n = n - n^*$ and

$$D = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}.$$

Candidate eigval / eig vec

$$③ \quad \begin{pmatrix} \Delta m(x,t) \\ \Delta n(x,t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cos \omega x$$

(eig.val.) (eigen frequency)

(here λ is eigenvalue, $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cos \omega x$ eigenvector, and ω is called the eigen frequency)

To satisfy the reflecting boundary conditions, we have

$$④ \quad \boxed{\omega = \frac{k\pi}{L}} \qquad (k = 0, 1, 2, \ldots)$$

Substitution of ③ into ②:

$$(\lambda I - A + \omega^2 D)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

Existence of non trivial solution (α, β) requires that $\det(\lambda I - A + \omega D) = 0$, i.e.,

$$⑤ \quad \boxed{\begin{aligned} &\lambda^2 + p(\omega)\lambda + q(\omega) = 0 \\ &p(\omega) = \omega^2(\mu + \nu) - \mathrm{tr} A \\ &q(\omega) = \omega^2(\omega^2 \mu\nu - \mu a_{22} - \nu a_{11}) + \det A \end{aligned}}$$

(characteristic equation).

model ① allows for diffusion instability whenever ⑤ has a solution (λ, ω) with $\mathrm{Re}\,\lambda > 0$ and $\omega = \frac{k\pi}{L}$ for some integer $k \geqslant 1$.

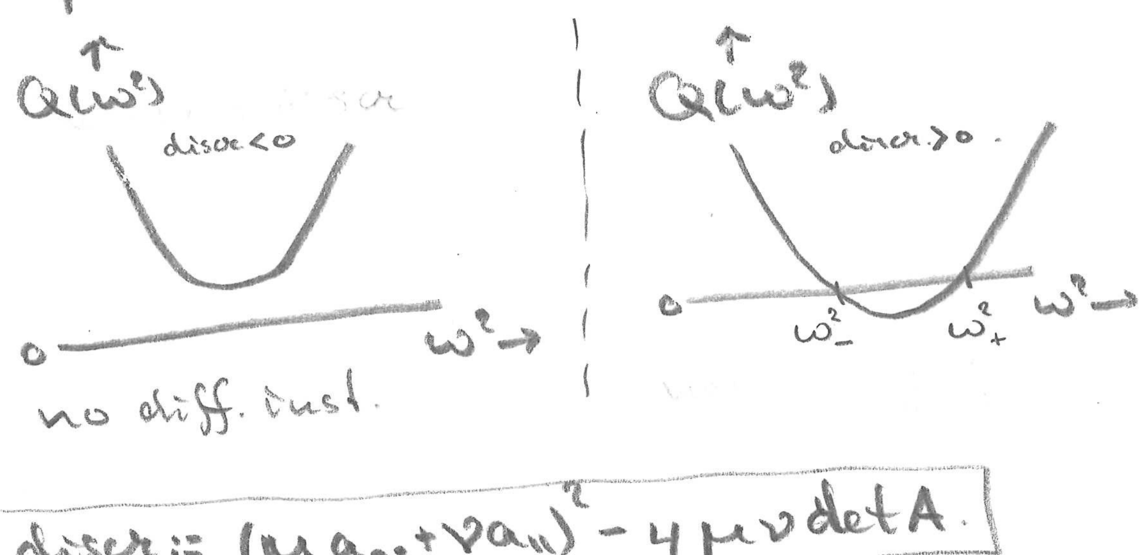
Since by assumption $\mathrm{tr} A < 0$, we have $p(\omega) > 0$. And so,

$\mathrm{Re}\,\lambda > 0 \iff q(\omega) < 0$.

Write $q(\omega) = Q(\omega^2)$ where

$$Q(\omega^2) = \omega^2(\omega^2 \mu\nu - \mu a_{22} - \nu a_{11}) + \det A$$

which is a 2nd-order polynomial in ω^2 representing a valley parabola



$\boxed{\text{discr} := (\mu a_{22} + \nu a_{11})^2 - 4\mu\nu \det A.}$

model ① allows for diffusion instability whenever ⑤ has a solution (λ, ω) with $\operatorname{Re}\lambda > 0$ and $\omega_- < \omega = \frac{k\pi}{L} < \omega_+$ for some integer $k \geq 1$.

$$\left\{ \begin{array}{l} \omega_\pm^2 := \frac{1}{2\mu\nu}\left(\mu a_{22} + \nu a_{11} \pm \sqrt{\mathrm{discr}}\right) > 0 \\ \mathrm{discr} := (\mu a_{22} + \nu a_{11})^2 - 4\mu\nu \det A > 0 \end{array} \right.$$

which is satisfied if and only if

$$\boxed{\mu a_{22} + \nu a_{11} > 2\sqrt{\mu\nu \det A} > 0}$$

(Since by assumption $\det A > 0$, the inequality on the right is always satisfied)

Rewrite as

$$\boxed{\sqrt{\frac{\mu}{\nu}} \cdot a_{22} + \sqrt{\frac{\nu}{\mu}}\, a_{11} > 2\sqrt{\det A}}$$

Conclusion

Suppose

① $\operatorname{tr} A < 0$ | non-spatial hyperbolic stability

② $\det A > 0$

③ $\sqrt{\frac{\mu}{\nu}}\, a_{22} + \sqrt{\frac{\nu}{\mu}}\, a_{11} > 2\sqrt{\det A}$

④ $\exists\, k \in \mathbb{N}_+ : \ \omega_- < \frac{k\pi}{L} < \omega_+$

where

$$\begin{cases} \omega_\pm = \frac{1}{2\mu\nu}\left(\mu a_{22} + \nu a_{11} \pm \sqrt{\text{discr.}}\right) \\ \text{discr.} = (\mu a_{22} + \nu a_{11})^2 - 4\mu\nu \det A. \end{cases}$$

- Then there is an eigenvalue λ with $\underline{\operatorname{Re}\lambda > 0}$ and with corresponding _positive_ eigenfrequency $\omega \in (\omega_-, \omega_+)$, and hence, model ① allows for diffusion instability.

Interpretation:

①, ② and ③ imply that

$\boxed{a_{11} a_{22} < 0}$ and $\boxed{a_{12} a_{21} < 0}$

This leaves us with two basic sign-structures of the Jacobimatrix A:

$A = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$ "activator-inhibitor" system

and

$A = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$ "positive feedback" system.

Suppose that $A = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$.

Then $\underbrace{a_{11}}_{>0} + \underbrace{a_{22}}_{<0} \frac{\mu}{\nu} > 0$, i.e.,

$\boxed{\frac{\mu}{|a_{11}|} < \frac{\nu}{|a_{22}|}}$ which means

that the activator m should have a relative low diffusion relative to the inhibitor n.
