MATHEMATICAL MODELLING

HOMEWORK SOLUTIONS

September 30, 2015

Exercise 8

Consider the model *before scaling*

$$\frac{dX}{dt} = \lambda(X_{\max} - X)Y - \delta X$$
$$\frac{dY}{dt} = -\lambda(X_{\max} - X)Y + \beta X - \mu Y.$$

(a) Let $\varepsilon > 0$ be a small scaling parameter and substitute $\lambda = \lambda_a/\varepsilon$ and $Y = \varepsilon Y_a$ (so that λ_a has the same order of magnitude of the other parameters, and Y_a has the same order of magnitude of X). The new system is

$$\frac{dX}{dt} = \frac{\lambda_a}{\varepsilon}\varepsilon(X_{\max} - X)Y_a - \delta X = \lambda_a(X_{\max} - X)Y_a - \delta X \tag{1}$$
$$\varepsilon\frac{dY_a}{dt} = -\frac{\lambda_a}{\varepsilon}\varepsilon(X_{\max} - X)Y_a + \beta X - \varepsilon\mu Y_a = -\lambda_a(X_{\max} - X)Y_a + \beta X - \varepsilon\mu Y_a. \tag{2}$$

From the previous system we can see that X is a slow variable and Y_a is a fast variable (dY_a/dt is very large compared to dX/dt).

Before studying the slow dynamics, we check that the fast system admits a stable equilibrium. If this is not true, the slow manifold is not well defined. To describe the fast dynamics, we introduce the fast time $\tau := t/\varepsilon$ and consider the system

$$\frac{dX}{d\tau} = \frac{dX}{dt}\frac{dt}{d\tau} = \varepsilon\left[\lambda_a(X_{\max} - X)Y_a - \delta X\right]$$
$$\frac{dY_a}{d\tau} = \frac{dY_a}{dt}\frac{dt}{d\tau} = \varepsilon\frac{1}{\varepsilon}\left[-\lambda_a(X_{\max} - X)Y_a + \beta X - \varepsilon\mu Y_a\right].$$

In the limit $\varepsilon \to 0$, we have $\frac{dX}{d\tau} = 0$, therefore the slow variable is constant and the fast dynamics is described by the 1-dimensional ODE

$$\frac{dY_a}{d\tau} = -\lambda_a(X_{\max} - X)Y_a + \beta X.$$

For fixed $X \leq X_{\max}$, the fast dynamics has a stable equilibrium

$$\hat{Y}_a = \frac{\beta X}{\lambda_a(X_{\max} - X)} \tag{3}$$

(to check it, you can plot dY_a/dt as a function of Y_a, observe that it is a straight line with negative slope, and therefore $dY_a/dt > 0$ if $Y_a < \hat{Y}_a$, $dY_a/dt < 0$ otherwise). Therefore, the slow manifold is well defined and it makes sense to study the slow dynamics.

We can obtain the equations for the slow dynamics by taking the limit of system (1)–(2) as $\varepsilon \to 0$,

$$\begin{aligned}\frac{dX}{dt} &= \lambda_a(X_{\max} - X)Y_a - \delta X \\ 0 &= -\lambda_a(X_{\max} - X)Y_a + \beta X.\end{aligned}$$

Observe that the slow manifold is obtained by solving the second equation, and it coincides with the equilibrium of the fast dynamics (3). By substituting (3) in the first equation and noting that (3) has to be positive, we get the following equation for the slow dynamics

$$\frac{dX}{dt} = \begin{cases}(\beta - \delta)X & \text{if } X \le X_{\max} \\ 0 & \text{otherwise.}\end{cases}$$

(b) Let $\varepsilon > 0$ be a small scaling parameter and consider $Y_b = \varepsilon Y$, and $\delta_b = \varepsilon\delta$, $\beta_b = \varepsilon\beta$ (so that Y_b, δ_b, β_b have the same order of magnitude of the other variables and parameters). The system becomes

$$\begin{aligned}\frac{dX}{dt} &= \lambda(X_{\max} - X)\frac{Y_b}{\varepsilon} - \frac{\delta_b}{\varepsilon}X \\ \frac{1}{\varepsilon}\frac{dY_b}{dt} &= -\lambda(X_{\max} - X)\frac{Y_b}{\varepsilon} + \frac{\beta_b}{\varepsilon}X - \mu\frac{Y_b}{\varepsilon}\end{aligned}$$

and, by simplifying, we get

$$\begin{aligned}\varepsilon\frac{dX}{dt} &= \lambda(X_{\max} - X)Y_b - \delta_b X \\ \frac{dY_b}{dt} &= -\lambda(X_{\max} - X)Y_b + \beta_b X - \mu Y_b.\end{aligned}$$

This is a slow-fast system because dX/dt is much larger than dY_b/dt, hence in this case X is the fast variable.

As before, we study the fast dynamics first, to make sure that the slow manifold is well-defined. We introduce the fast time $\tau := t/\varepsilon$ and consider

$$\begin{aligned}\frac{dX}{d\tau} &= \frac{dX}{dt}\frac{dt}{d\tau} = \varepsilon\frac{1}{\varepsilon}\left[\lambda(X_{\max} - X)Y_b - \delta_b X\right] \\ \frac{dY_b}{d\tau} &= \frac{dY_b}{dt}\frac{dt}{d\tau} = \varepsilon\left[-\lambda(X_{\max} - X)Y_b + \beta_b X - \mu Y_b\right].\end{aligned}$$

In the limit $\varepsilon \to 0$, Y_b is constant and the equation for the fast dynamics is

$$\frac{dX}{d\tau} = \lambda(X_{\max} - X)Y_b - \delta_b X = -(\lambda Y_b + \delta_b)X + \lambda X_{\max}Y_b,$$

The equilibrium is

$$\hat{X} = \frac{\lambda X_{\max}Y_b}{\lambda Y_b + \delta_b} < X_{\max}, \tag{4}$$

and it easy to verify stability because the right-hand side is a straight line with a negative slope with respect to X. Hence, the slow manifold is well defined.

The equation for the slow dynamics are obtained by taking the limit as $\varepsilon \to 0$:

$$\begin{aligned} 0 &= \lambda(X_{\max} - X)Y_b - \delta_b X \\ \frac{dY_b}{dt} &= -\lambda(X_{\max} - X)Y_b + \beta_b X - \mu Y_b. \end{aligned}$$

The slow manifold is obtained by solving the second equation and is exactly (4), and by substituting into the first equation, we get the following equation for the slow dynamics

$$\frac{dY_b}{dt} = (\beta_b - \delta_b)\frac{\lambda X_{\max} Y_b}{\lambda Y_b + \delta_b} - \mu Y_b = \frac{Y_b}{\lambda Y_b + \delta_b}\left[\lambda X_{\max}(\beta_b - \delta_b) - \mu\delta_b - \mu\lambda Y_b\right]$$

which may be written as

$$\frac{dY_b}{dt} = \frac{1}{\lambda Y_b + \delta_b} r Y_b \left(1 - \frac{Y_b}{K}\right)$$

with

$$r = \lambda X_{\max}(\beta_b - \delta_b) - \mu\delta_b, \qquad K = \frac{\lambda X_{\max}(\beta_b - \delta_b) - \mu\delta_b}{\mu\lambda}.$$

Remark 1. By comparing this exercise and the example in class (where we obtained a logistic equation for the slow dynamics), we can note how a different scaling of the parameters lead to very different dynamical behaviour.

Exercise 9

It is important to note that the exercise does not have a unique possible solution. Different choice of scaling lead to different results. Some choices may lead to uninteresting problems (e.g., both populations go extinct) or problems that cannot be analysed with the slow-fast method (e.g., if the fast dynamics does not have a stable equilibrium). Once you have chosen the parameter scaling, the analysis consists of the same steps. Note that, in order to carry of the stability analysis of the fast system, you will need to specify the growth function $f(X)$ (for instance, linear growth or logistic growth).

Some possible scaling choices.
(a) $\alpha = \frac{\alpha_0}{\varepsilon},\ \beta = \frac{\beta_0}{\varepsilon},\ T = T_0\varepsilon,\ p = p_0\varepsilon$;
(b) $\alpha = \frac{\alpha_0}{\varepsilon},\ \beta = \frac{\beta_0}{\varepsilon}$;
(c) $\beta = \frac{\beta_0}{\varepsilon}$;
(d) $p = \varepsilon p_0,\ \delta = \varepsilon\delta_0$ (note that, in this scaling, t is already the fast time).

Observe how some choices lead to a trivial dynamics, in the sense that both populations go extinct (prey dies fast, predator dies slowly). It's much more interesting if you find a nontrivial dynamics, where coexistence is possible!

Recipe for fast-slow analysis. Here I summarise the main steps of a fast-slow analysis, that you can apply to your specific scaling by following the scheme of Exercise 8.

1. **Fast-slow system.** Consider

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = \frac{1}{\varepsilon} g(x, y)$$

where $\varepsilon > 0$ is a small, dimensionless scaling parameter.

$$\dot{y} \text{ is large } \Rightarrow \quad y \text{ fast variable}$$
$$x \text{ slow variable}$$

t is the *slow time*.

Before analysing the slow dynamics, we need to check that the fast dynamics has a stable equilibrium.

If no unique stable equilibrium exists, then the slow manifold is not well defined.

2. **Fast dynamics.** Introduce the *fast time* $\tau := \frac{t}{\varepsilon}$.

The system describing the fast dynamics is

$$\frac{dx}{d\tau} = \varepsilon f(x, y)$$
$$\frac{dy}{d\tau} = g(x, y). \tag{5}$$

In the limit $\varepsilon \to 0$ the first equation gives $dx/d\tau = 0$. Indeed, if we look in fast time, the x variable is changing so slowly that we don't recognise any change. It is fixed at a certain value x.

We do phase-plane analysis or linear stability analysis to check that equation (5) has a stable equilibrium $\hat{y} = \hat{y}(x)$ satisfying

$$0 = g(x, \hat{y}).$$

This means that, when we look at slow time, for any value of x the fast variable y goes "instantly" to its equilibrium value $\hat{y}(x)$.

3. **Slow dynamics.** If the fast dynamics admits a unique stable equilibrium $\hat{y}$, then we look back at the dynamics in slow time

$$\frac{dx}{dt} = f(x, y)$$
$$\varepsilon \frac{dy}{dt} = g(x, y).$$

In the limit $\varepsilon \to 0$, the second equation $0 = g(x, \hat{y})$ defines the slow manifold $\hat{y}(x)$. Observe that this is the same condition defining the stable equilibrium of the fast dynamics.

We describe the dynamics on the slow manifold by plugging this into the first equation,

$$\frac{dx}{dt} = f(x, \hat{y}(x)),$$

which is an ODE in the variable x. We can study the slow dynamics using phase-plane and linear stability analysis in the normal way.