MATHEMATICAL MODELLING

HOMEWORK SOLUTIONS

September 23, 2015

Exercise 6

(a) An individual in a site (e.g., plant or flower) produces $n+1$ offsprings (e.g., seeds) that disperse in the environment. After reproducing, the mother dies and her site becomes free.

(b) System of ODEs

$$\begin{aligned} \frac{dX}{dt} &= \quad -\beta X \quad -\delta X + \lambda SY \\ \frac{dS}{dt} &= \quad +\beta X \quad +\delta X - \lambda SY \\ \frac{dY}{dt} &= +(n+1)\beta X \quad - \lambda SY - \mu Y \end{aligned} \tag{1}$$

(c) The total density of sites $s_0 = X + S$ satisfies $\frac{d}{dt}(X+S) = 0$, therefore it is constant in time. Then we can write $S = s_0 - X$ and, by substituting into (1), we get the system of two ODEs

$$\begin{aligned} \frac{dX}{dt} &= -(\beta+\delta)X + \lambda s_0 Y - \lambda XY \\ \frac{dY}{dt} &= (n+1)\beta X - (\mu + \lambda s_0)Y + \lambda XY \end{aligned}$$

(d) We assume that the number of seeds produced is very large, and that the mortality rate of seeds without a site is very high, and we do that by substituting the parameters n and μ by n/ε, μ/ε, where $\varepsilon > 0$ is a small scaling parameter. Therefore, we study the fast-slow system

$$\begin{aligned} \frac{dX}{dt} &= -(\beta+\delta)X + \lambda s_0 Y - \lambda XY \\ \frac{dY}{dt} &= \left(\frac{n}{\varepsilon}+1\right)\beta X - \left(\frac{\mu}{\varepsilon} + \lambda s_0\right) Y + \lambda XY \end{aligned} \tag{2}$$

(e) In particular, the second equation becomes

$$\varepsilon \frac{dY}{dt} = n\beta X + \varepsilon\beta X - \mu Y - \varepsilon\lambda s_0 Y + \varepsilon\lambda XY$$

and in the limit $\varepsilon \to 0$, system (2) reduces to

$$\begin{aligned} \frac{dX}{dt} &= -(\beta+\delta)X + \lambda s_0 Y - \lambda XY \\ 0 &= n\beta X - \mu Y \end{aligned} \tag{3}$$

By solving the second equation, we obtain the *slow manifold*

$$Y = \frac{n\beta}{\mu}X.$$

By plugging this into the first equation of system (3), we obtain the single ODE on the slow manifold

$$\frac{dX}{dt} = -(\beta+\delta)X + \lambda s_0 \frac{n\beta}{\mu}X - \lambda\frac{n\beta}{\mu}X^2 = rX\left(1-\frac{X}{K}\right),$$

where

$$r = \lambda s_0 \frac{n\beta}{\mu} - \beta - \delta, \qquad K = \frac{\lambda s_0 n\beta - (\beta+\delta)\mu}{\lambda n\beta}.$$

Remark 1. This exercise combines two different methods that are useful to reduce the dimension of the original model: *conservation law* and *time-scale separation*. These two methods allowed us to go from a system of 3 equations to a single equation.

Exercise 7

i-states & transitions	*Interpretation*
$\boxed{S}$	healthy but susceptible individual
$\boxed{I}$	infected individual
$\boxed{R}$	recovered and temporally immune individual
$\boxed{S}+\boxed{I} \xrightarrow{\beta} \boxed{I}+\boxed{I}$	a susceptible gets in contact with an infected and becomes infected
$\boxed{I} \xrightarrow{\gamma} \boxed{R}$	infected recovers from the disease and becomes immune
$\boxed{R} \xrightarrow{\delta} \boxed{S}$	recovered loses immunity and becomes susceptible again

This kind of models are called *SIR models* (from the letters denoting the different classes).

Remark 2. Observe that demography is not included in the model, in the sense that natural birth or death of individuals are ignored. This is the case if the infection period is short compared to the lifetime of an individual (e.g., flue epidemic). Moreover, since one can exit the infected class only through recovery, this is not a deadly disease.

From the previous remark, since there are no death/births in the model, the total population density $N = s + i + r$ is constant in time, and indeed it is easy to check that

$\frac{dN}{dt} = 0$. We use the conservation relation for reducing the dimension of the model, by substituting $r = N - s - i$. Therefore,

$$\begin{aligned} \frac{ds}{dt} &= -\beta si + \delta(N - s - i) \\ \frac{di}{dt} &= +\beta si - \gamma i \end{aligned} \tag{4}$$

Phase-plane analysis.

1. From biological assumptions, we consider the plane $s, i \geq 0$.
2. s-isocline: $i = \dfrac{\delta(N - s)}{\beta s + \delta}$
   i-isocline: $i = 0$ or $s = \dfrac{\gamma}{\beta}$ If we study the sign of the derivative, we observe that

$$\begin{aligned} \frac{ds}{dt} > 0 &\Leftrightarrow i < \frac{\delta(N - s)}{\beta s + \delta} \\ \frac{di}{dt} > 0 &\Leftrightarrow s > \frac{\gamma}{\beta}. \end{aligned}$$

3. The s-isocline is a hyperbola intersecting the axes in $(0, \frac{\delta N}{\beta})$ and $(N, 0)$.

We split two cases: $N < \frac{\gamma}{\beta}$ and $N \geq \frac{\gamma}{\beta}$.

First case: $N < \frac{\gamma}{\beta}$. Compare Figure 1.

4. The only equilibrium is $E_1 = (N, 0)$
5. All the arrows are pointing towards the equilibrium: E_1 is stable.

Second case: $N \geq \frac{\gamma}{\beta}$. Compare Figure 2.

4. The equilibria are $E_1 = (N, 0)$ and

$$E_2 = (\bar{s}, \bar{i}) = \left(\frac{\gamma}{\beta}, \frac{\delta(\beta N - \gamma)}{\beta(\gamma + \delta)} \right).$$

5. Looking at the arrows, we observe that E_1 is unstable (saddle), but we cannot conclude about the stability of E_2.
6. To investigate the stability of E_2, we need linear stability analysis.

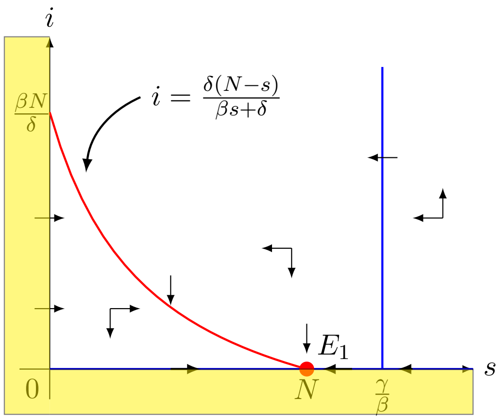

Figure 1: phase portrait of case $N < \frac{\gamma}{\beta}$.

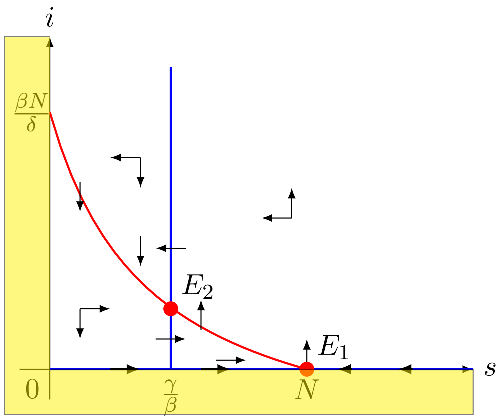

Figure 2: phase portrait of case $N \geq \frac{\gamma}{\beta}$.