# MATHEMATICAL MODELLING

HOMEWORK SOLUTIONS

September 16, 2015

## Exercise 4

**Interpretation.** I propose two possible interpretations: a consumer–resource model with constant influx $\varphi$ of resource (you can see it also as a predator–prey model, where the prey is the resource), or an epidemiological model (SIS model) with constant immigration influx $\varphi$ of susceptible individuals.

*Resource–consumer*: the resource is eaten at a rate $\beta$. With probability $p$, the resource is converted into one offspring. With probability $1-p$, there is no reproduction at all. The resource does not reproduce itself: the only influx is the forcing term $\varphi$.

*Epidemics*: if a susceptible meets an infected, she will be infected herself at a rate $\gamma = \beta$. There is no recovery from the disease (or, alternatively, recovered individuals are immune to the disease, so they are not susceptible any more). Infected individuals might have a higher death rate than susceptibles: $\delta \geq \alpha$.

| *i-states & transitions* | *Resource–consumer model* | *Epidemiological model* |
|---|---|---|
| $\boxed{X}$ | resource | susceptible individual |
| $\boxed{Y}$ | consumer | infected individual |
| $\emptyset \xrightarrow{\varphi} \boxed{B}$ | resource influx | immigration of healthy ind |
| $\boxed{Y} + \boxed{X} \xrightarrow{(1-p)\beta} \boxed{Y}$ | consumption and reproduction, $\gamma = p\beta$ | infection (with $p = 0, \gamma = \beta$) |
| $\boxed{Y} + \boxed{X} \xrightarrow{p\beta} 2\,\boxed{Y}$ | | |
| $\boxed{X} \xrightarrow{\alpha} \dagger$ | natural decay of resource | natural death of healthy ind |
| $\boxed{Y} \xrightarrow{\delta} \dagger$ | natural death of consumer | death of infected individual |

The ODE system is

$$\begin{cases} \dfrac{dx}{dt} = \varphi - \alpha x - \beta x y \\ \dfrac{dy}{dt} = \gamma x y - \delta y \end{cases}$$

**Phase-plane analysis.** Recipe for drawing a phase-plane picture:

1. remember conditions from biological assumption, e.g., $x \geq 0$, $y \geq 0$;
2. compute isoclines $\dot{x} = 0$, $\dot{y} = 0$;
3. plot the isoclines in the plane $(x, y)$ (tip: plot $x$- and $y$-isoclines in different colours), and the horizontal (no change in $y$, $\dot{y} = 0$) or vertical (no change in $x$, $\dot{x} = 0$) arrows on the isoclines; maybe you need to consider different cases?
4. equilibrium points are points where different isoclines (i.e., different colours!) intersect;
5. study where $\dot{x} > 0$ and $\dot{y} > 0$ and plot the corresponding arrows in all the different regions of the plane;
6. observe the direction of the flow, especially in correspondence of equilibria, and deduce stability:
   (i) all the arrows are pointing towards the equilibrium $\Rightarrow$ stable;
   (ii) at least one arrow is pointing outwards $\Rightarrow$ unstable;
   (iii) the arrows are turning around $\Rightarrow$ cannot conclude, need deeper invastigation (we will study *linear stability analysis*: eigenvalues of the Jacobian matrix).

From biological assumptions, we are interested in $x \geq 0$, $y \geq 0$. The isoclines are

$$\dot{x} = 0 \qquad \Leftrightarrow \qquad y = \frac{\varphi}{\beta x} - \frac{\alpha}{\beta}$$

$$\dot{y} = 0 \qquad \Leftrightarrow \qquad y = 0 \text{ or } x = \frac{\delta}{\gamma}$$

The $x$-isocline is a hyperbola asymptotic to the $y$ axis and intersecting the $x$ axis in $\left(\frac{\varphi}{\alpha}, 0\right)$. The $y$-isocline is the union of two straight lines parallel to the axis.

In order to being able to plot the isoclines and their intersection, we need to separate two different cases: $\frac{\varphi}{\alpha} \leq \frac{\delta}{\gamma}$ or $\frac{\varphi}{\alpha} > \frac{\delta}{\gamma}$ (see Figure 1 and 2). We draw the isoclines and the arrows observing that

$$\dot{x} > 0 \qquad \Leftrightarrow \qquad y < \frac{\varphi}{\beta x} - \frac{\alpha}{\beta}$$

$$\dot{y} > 0 \qquad \Leftrightarrow \qquad x > \frac{\delta}{\gamma}$$

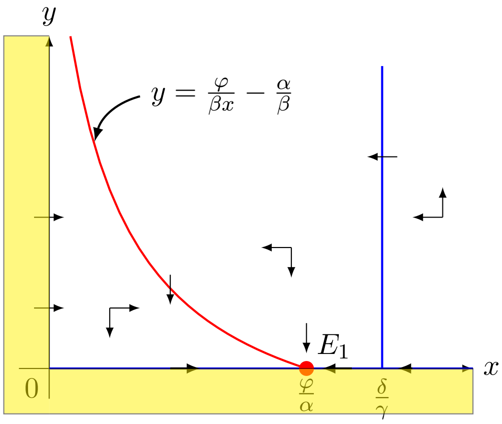


**Figure 1:** isoclines in the case $\frac{\varphi}{\alpha} \leq \frac{\delta}{\gamma}$.

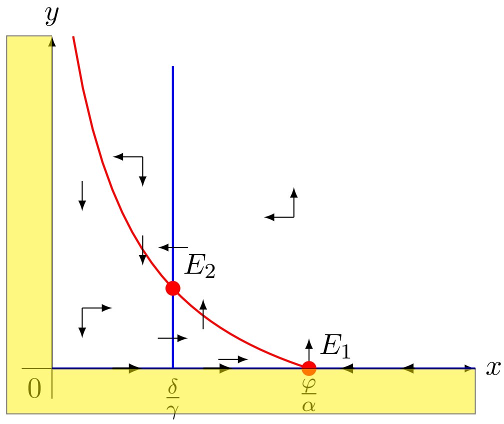


**Figure 2:** isoclines in the case $\frac{\varphi}{\alpha} > \frac{\delta}{\gamma}$.

First case: $\frac{\varphi}{\alpha} \leq \frac{\delta}{\gamma}$ (Figure 1): the only equilibrium point (intersection between different isoclines) is $E_1 = (\frac{\varphi}{\alpha}, 0)$. Observing the direction of the flow (arrows), we conclude that $E_1$ is stable (all arrows pointing towards it).

Second case: $\frac{\varphi}{\alpha} > \frac{\delta}{\gamma}$ (Figure 2):there are two intersection points between different isoclines,

$$E_1 = \left(\frac{\varphi}{\alpha}, 0\right) \qquad \text{and} \qquad E_2 = \left(\frac{\delta}{\gamma}, \frac{\varphi\gamma - \alpha\delta}{\beta\delta}\right)$$

By observing the direction of the arrows, we can say that $E_1$ is unstable (one arrow pointing outwards), but we cannot conclude anything about the stability of $E_2$, because the arrows are turning around.

**Remark 1.** Be careful: the point $(\frac{\delta}{\gamma}, 0)$ is *not* an equilibrium of the system, because it is given by the intersection of isoclines relative to the *same* variable (blue isoclines in the picture). Indeed, in that point we have $\dot{x} \neq 0$.

## Exercise 5

***i*-states.** After some trials, I think the best thing to do for representing the dynamics of larvae is to define different $i$-states for larvae with $0, 1$ and $2$ or more eggs. Therefore, we consider the following $i$-states: butterfly $\boxed{B}$, parasitoid $\boxed{P}$, larva without parasitoid eggs $\boxed{L_0}$, larva with one single egg $\boxed{L_1}$, larva with two or more eggs $\boxed{L_2}$.

To make the model even simpler, I assume that larvae with two or more eggs die instantly, and avoid to include the state $\boxed{L_2}$.

**$i$-state transitions.**

| | | |
|---|---|---|
| $(a)$ | $\boxed{B} \xrightarrow{\beta} \boxed{L_0} + \boxed{B}$ | production of larvae |
| $(b)$ | $\boxed{P} + \boxed{L_0} \xrightarrow{\lambda_0} \boxed{P} + \boxed{L_1}$ | encounter and egg deposition in empty larva |
| $(c)$ | $\boxed{P} + \boxed{L_1} \xrightarrow{\lambda_1} \boxed{P} + \dagger$ | encounter and egg deposition in larva with 1 egg |
| $(d)$ | $\boxed{L_0} \xrightarrow{\gamma_0} \boxed{B}$ | maturation of a larva into butterfly |
| $(e)$ | $\boxed{L_1} \xrightarrow{\gamma_1} \boxed{P}$ | maturation of a larva into parasitoiod |
| $(f1)$ | $\boxed{B} \xrightarrow{\delta_B} \dagger$ | mortality of butterfly |
| $(f2)$ | $\boxed{P} \xrightarrow{\delta_P} \dagger$ | mortality of parasitoid |
| $(f3)$ | $\boxed{L_0} \xrightarrow{\delta_0} \dagger$ | mortality of larva |
| $(f4)$ | $\boxed{L_1} \xrightarrow{\delta_1} \dagger$ | mortality of larva with one egg |

**Remark 2.** Some comments about the rates:
(i) the encounter rates $\lambda_0, \lambda_1$ in processes $(b)$ and $(c)$ may depend for instance on the place where the larva has been produced, on the motility of larvae and of parasites or on hiding ability of larva. Note that, if such properties remain unchanged;
(ii) the development rate of larvae with 0 or 1 egg might be different (e.g., if the development of parasitoid requires less resource). Therefore, we allow different maturation rates $\gamma_0, \gamma_1$ in processes $(d)$ and $(e)$. Note that, if the development property remains exactly the same, than the rates would also be the equal. The same remark applies to the mortality rates in processes $(f3)$ and $(f4)$.

**System of ODEs for population densities.**

$$
\begin{array}{rcccccccc}
 & & (a) & (b) & (c) & (d) & (e) & (f) \\
\dfrac{dB}{dt} & = & & & & +\gamma_0 L_0 & & -\delta_B B \\
\dfrac{dP}{dt} & = & & & & & +\gamma_1 L_1 & -\delta_P P \\
\dfrac{dL_0}{dt} & = & +\beta B & -\lambda_0 P L_0 & & -\gamma_0 L_0 & & -\delta_0 L_0 \\
\dfrac{dL_1}{dt} & = & & +\lambda_0 P L_0 & -\lambda_1 P L_1 & & -\gamma_1 L_1 & -\delta_1 L_1
\end{array}
$$