MATHEMATICAL MODELLING

HOMEWORK SOLUTIONS

November 18, 2015

I will solve Exercise 21 step-by-step and then provide the results for Exercise 19 and 20. I hope you can reproduce the derivation yourself.

Exercise 21

(a) Type of movement: diffusion with constant coefficient and positive taxis ($J_{\text{taxis}} = an\partial_x n$). For instance, animals in a reservoir with fences around, such that they cannot go out and they tend to move towards high densities of individuals (positive taxis).

(b) Consider a constant equilibrium $n(x) = \overline{n} < \frac{D}{a}$.

Remark 1. You can verify that every positive constant $\overline{n}$ is an equilibrium of the system. We can choose a particular one, for instance, by normalizing the total population density $\int_0^L \overline{n}dx$ or the average population density $\frac{1}{L}\int_0^L \overline{n}dx$.

To linearize the equation, we consider a perturbation $u(x,t)$ that is $\mathcal{C}^2$-small, in the sense that $|u(x,t)|$, $|\partial_x u(x,t)|$ and $|\partial_{xx} u(x,t)|$ are uniformly small (say $o(\varepsilon)$) in the domain $x \in [0, L]$, for all $t \geq 0$.

Substitute $n(x,t) = \overline{n} + u(x,t)$ into the original problem.

$$\begin{cases} \partial_t u = D\partial_{xx}u - a\partial_x((\overline{n}+u)\partial_x u) = D\partial_{xx}u - a\overline{n}\partial_{xx}u - a\underbrace{\left((\partial_x u)^2 + u\partial_{xx}u\right)}_{o(\varepsilon^2)} \\ 0 = D\partial_x u - a(\overline{n}+u)\partial_x u) = D\partial_x u - a\overline{n}\partial_x u - \underbrace{au\partial_x u}_{o(\varepsilon^2)} \qquad \text{for } x = 0, L. \end{cases}$$

Since $u(x,t)$ is $\mathcal{C}^2-$small, the terms $(\partial_x u)^2$, $u\partial_{xx}u$, and $u\partial_x u$ are $o(\varepsilon^2)$, so they are negligible. Therefore, the linearized system is

$$\begin{cases} \partial_t u = (D - a\overline{n})\,\partial_{xx}u \\ (D - a\overline{n})\,\partial_x u = 0 \qquad \text{for } x = 0, L. \end{cases}$$

Remember that we assume $D - a\overline{n} > 0$. We now take a trial solution of the form $u(x,t) = v(x)e^{\lambda t}$ with $\lambda, v(x) \in \mathbb{C}$, and plug into the linearized system to obtain the characteristic equation for λ,

$$\begin{cases} \lambda v(x) = (D - a\overline{n})\,v''(x) \\ (D - a\overline{n})\,v'(x) = 0 \qquad \text{for } x = 0, L. \end{cases} \tag{1}$$

The first equation is a second order linear ODE with constant coefficients,

$$(D - a\overline{n})\, v''(x) - \lambda v(x) = 0.$$

We solve it by the characteristic polynomial

$$(D - a\overline{n})\, y^2 - \lambda = 0 \quad \Leftrightarrow \quad y_{1,2} = \pm\sqrt{\frac{\lambda}{D - a\overline{n}}}.$$

Therefore, the solutions are

$$v(x) = Ae^{y_1 x} + Be^{y_2 x}, \quad A, B \in \mathbb{R}. \tag{2}$$

To determine A and B, we impose the boundary conditions:

$$(D - a\overline{n})\, v'(x) = 0 \text{ at } x = 0, L \quad \Leftrightarrow \begin{cases} (D - a\overline{n})\,(Ay_1 + By_2) = 0 \\ (D - a\overline{n})\left(Ay_1 e^{y_1 L} + By_2 e^{y_2 L}\right) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} By_2 = -Ay_1 \\ e^{y_1 L} = e^{y_2 L} \end{cases} \tag{3}$$

We want to solve the system for $\lambda \in \mathbb{C}$ (remember that y_1, y_2 depend on λ).

$$e^{y_1 L} = e^{y_2 L} \Leftrightarrow y_1 L = y_2 L + 2k\pi i, \quad \text{for some } k \in \mathbb{Z}$$

$$\Leftrightarrow L\sqrt{\frac{\lambda_k}{D - a\overline{n}}} = -L\sqrt{\frac{\lambda_k}{D - a\overline{n}}} + 2k\pi i \Leftrightarrow \sqrt{\frac{\lambda_k}{D - a\overline{n}}} = \frac{k\pi i}{L} \Leftrightarrow \lambda_k = -(D - a\overline{n})\frac{k^2\pi^2}{L^2}$$

Be careful: it might be that some λ_k correspond to $v(x) = 0$, and this is NOT a proper eigenfunction! In order to determine which λ_k are *true* eigenvalues, we should check that the corresponding eigenfunction $v_k(x)$ is nonzero. For a given λ_k, from (2) and (3) follows that

$$v_k(x) = A\left(e^{i\frac{k\pi}{L}x} + e^{-i\frac{k\pi}{L}x}\right) = \frac{A}{2}\cos\left(\frac{k\pi}{L}x\right)$$

The function $v_k(x)$ is nonzero for any $k \in \mathbb{R}$, therefore any λ_k can be an eigenvalue. BUT, for $k = 0$ we have $v(x) = \frac{A}{2}\cos 0 = \frac{A}{2}$ and this perturbation is not permissible because it changes the total population density, while in the original problem there is conservation of total population density. Therefore we exclude $\lambda = 0$ from the eigenvalues and, since all the other λ_k, $k \neq 0$ are real and negative, we conclude that the equilibrium $\overline{n}$ is stable.

(c) If $\overline{n} > \frac{D}{a}$ the previous analysis holds unchanged, and the eigenvalues are

$$\lambda_k = -(D - a\overline{n})\frac{k^2\pi^2}{L^2} = (a\overline{n} - D)\frac{k^2\pi^2}{L^2} > 0$$

for any $k \in \mathbb{Z}$. In this case all the eigenvalues are real and positive, therefore the equilibrium is unstable.

After Exercise 21, I would like to complete the

Scheme for local stability analysis of spatial systems

1. Linearize the problem around the equilibrium $\overline{n}(x)$: consider a perturbation $u(x,t)$ that is $\mathcal{C}^2-$small (i.e., $|u|$, $|\partial_x u|$ and $|\partial_{xx}u|$ are small) and plug $n(x,t) = \overline{n}(x) + u(x,t)$ into the original problem. Simplify the equation using the equilibrium conditions. Keep only terms of first order in $|u|$ and neglect higher order terms. Remember that the linearized equation is a *linear equation.* Remember also to linearize the boundary conditions.

2. Consider a trial solution of the form $u(x,t) = v(x)e^{\lambda t}$, where $\lambda \in \mathbb{C}$ is an eigenvalue and $v(x) \in \mathbb{C}$ is the corresponding eigenfunction, and substitute this into the linearized problem. You will get the *characteristic equation* for λ and $v(x)$.

3. Solve the characteristic equation for λ. Determine the true eigenvalues by checking that the corresponding eigenfunction $v(x)$ is nonzero. If there is conservation of the total density in the original system, be sure that also the perturbations must conserve the total density. Perturbations such that $\int_0^L v(x)dx \neq 0$ are not proper eigenfunctions.

   The equilibrium $\overline{n}(x)$ is stable if all the eigenvalues have negative real part; it is unstable if there exists at least one eigenvalue with positive real part; if the maximum real part is exactly equal to zero, it is not possible to decide about stability.

Exercise 19

(a) Exponential growth, so we can think at a bacteria population that diffuses in a region and it is removed at the boundary. Maybe some mould or fungus on a delimited substrate region.

(b) $n = 0$ is an equilibrium. Linearization around $n = 0$ gives

$$\begin{cases} \partial_t u = au + D\partial_{xx}u \\ u = 0 & \text{for } x = 0, L. \end{cases}$$

Trial solution $v(x)e^{\lambda t}$ gives the characteristic equation

$$\begin{cases} \lambda v(x) = av(x) + Dv''(x) \\ v(x) = 0 & \text{for } x = 0, L. \end{cases}$$

The eigenvalues are

$$\lambda_k = a - \frac{Dk^2\pi^2}{L^2}, \quad k = 1, 2, \ldots$$

with corresponding eigenfunctions

$$v_k(x) = A\left(e^{\frac{ik\pi}{L}x} - e^{-\frac{ik\pi}{L}x}\right) = -2Ai\sin\left(\frac{k\pi}{L}x\right).$$

Notice that $v_0(x) = 0$, hence $k = 0$ does not give a proper eigenvalue: we take $k \geq 1$. The eigenvalues are real and the dominant one is $\lambda_1 = a - \frac{D\pi^2}{L^2}$, therefore the equilibrium $n = 0$ is stable if $a < \frac{D\pi^2}{L^2}$, unstable if $a > \frac{D\pi^2}{L^2}$.

Exercise 20

(a) Logistic growth, so we can think at an animal or plant population. For instance, some trees in a region with a rock boundary (reflecting) and a grown field at the other boundary (men keep it clean from seeds: absorbing). Only diffusive movement present.

(b) $n = 0$ is an equilibrium. Linearization around $n = 0$ gives

$$\begin{cases} \partial_t u = ru + D\partial_{xx}u \\ u(0) = 0, \quad u'(L) = 0. \end{cases}$$

Trial solution $v(x)e^{\lambda t}$ gives the characteristic equation

$$\begin{cases} \lambda v(x) = rv(x) + Dv''(x) \\ v(x) = 0, \quad v'(L) = 0. \end{cases}$$

The eigenvalues are

$$\lambda_k = r - \frac{D\pi^2(2k+1)^2}{4L^2}, \quad k \in \mathbb{Z}, k \neq 0$$

with corresponding eigenfunctions

$$v_k(x) = A\left(e^{\frac{ik\pi}{L}x} - e^{-\frac{ik\pi}{L}x}\right) = -2Ai\sin\left(\frac{k\pi}{L}x\right).$$

Notice that $v_0(x) = 0$, hence $k = 0$ does not give a proper eigenvalue: we take $k \neq 0$. The eigenvalues are real and the dominant one is $\lambda_{-1} = r - \frac{D\pi^2}{4L^2}$, therefore the equilibrium $n = 0$ is stable if $r < \frac{D\pi^2}{4L^2}$, unstable if $r > \frac{D\pi^2}{4L^2}$.