MATHEMATICAL MODELLING

HOMEWORK SOLUTIONS

November 11, 2015

Exercise 17

(a) From the picture on the left, we observe that the cross section is perpendicular to the flux of the water (maybe the arrows are not completely exhaustive, but look at the "waves"!). Translated into the picture on the right, this means that there is no horizontal movement of the water, and therefore no preferred horizontal movement of the fish in the river: fish movement is random and described only by diffusion.

Denote $n(x,t)$ the fish density, $p(x,t)$ the predator density. The fish population is growing with natural growth $g(n)$ and it is moving only according to diffusion, therefore it is described by the PDE

$$\partial_t n(x,t) = g(n(x,t)) - \partial_x J_{\text{diff}}(x,t) = g(n(x,t)) + D\partial_{xx}n(x,t).$$

The predator is well-mixed in compartment C: the density is the same everywhere. Therefore we do not include space structure: $p(x,t) = p(t)$. Denote $f(n)$ the functional response of the predator. Anyway, note that predation happens only at the border between land and water and therefore only fish at $x = 0$ are eaten: their density is $n(0,t)$. In conclusion, the evolution of predator is described by the ODE

$$\dot{p}(t) = \gamma f(n(0,t))p(t).$$

(b) Let us consider the boundary conditions for fish. We assume reflecting boundary at $x = L$, therefore

$$\partial_x n(x,t) = 0 \quad \text{at } x = L. \tag{1}$$

(More precisely, a reflecting boundary condition means that we assume that the total flux is zero at $x = L$. Here, the only thing happening is diffusion, therefore the total flux is $J_{\text{tot}} = J_{\text{diff}} = -D\partial_x n$, and from the condition $J_{\text{tot}} = 0$ we get (1).)

At $x = 0$ the boundary is not reflecting: some fish is eaten by predators. In this case, the total flux should compensate the loss of fish due to predation:

$$J_{\text{tot}} = J_{\text{diff}} = -f(n(0,t))p(t).$$

Note the minus sign: fish is eaten at the left border, so the "compensating" flux must be negative. In conclusion, the boundary condition is

$$\partial_x n(x,t) = \frac{1}{D} f(n(0,t))p(t) \quad \text{at } x = 0.$$

Remark 1. If you want to derive more rigorously the boundary condition, you can derive it by studying the microscopic level: you split the individuals in compartments of width Δx, analyse all the fluxes and reactions, and derive the continuous equations by introducing the interpolating functions and by letting $\Delta x \to 0$, similarly to what we have done in the lectures.

(c) By neglecting spatial structure of the predator, we implicitly assume that the movement is *instantaneous* (infinite speed), while we assume a finite movement speed for fish. This may follow also by the assumption that the compartment C is a very narrow stripe along the river.

As a different example, imagine a predator population that moves very slowly. In this case, it is more likely that offspring are produced close to the place where the food is. Therefore, it is more likely that the predator population density is higher close to the river, and therefore the predation force is higher. By neglecting spatial structure of the predator, we neglect this effect.

(d) This is an open question and many ideas and models are possible. Consider spatial diffusion of the predator in the compartment C: consider the density $p(x,t)$ in $-M \leq x \leq 0$. Let us consider the example in point (c), where offspring are produced where the food is consumed, i.e., at the boundary $x = 0$. In this case the diffusion equation and the boundary conditions for the predator are

$$\begin{aligned}
&\partial_t p(x,t) = D_p \partial_{xx} p(x,t) \\
&\partial_x p(-M,t) = 0 \\
&J(0,t) = -D_p \partial_x p(0,t) = -\gamma p(0,t) n(0,t)
\end{aligned}$$

Note that in the situation considered in this exercise, where the spatial domains of fish and predator are separated by a boundary, there is no density gradient of fish in the predator domain, and vice-versa (simply because there is no fish at all in the predator domain). In this sense, predator–fish taxis is not present in the model. There are other ways of introducing taxis in the model, for instance by introducing extra i-states (predator carrying the fish, "panicked" fish, etc), that provoke a response to their density gradient.

Exercise 18

Consider the equations

$$\begin{aligned}
&\partial_t n(x,t) = -\partial_x \left(-D\partial_x n(x,t) + \alpha n(x,t)\right) \\
&0 = -D\partial_x n(x,t) + \alpha n(x,t) \quad \text{for } x = 0, x = L.
\end{aligned} \tag{2}$$

(a) The total flux is $J_{\text{tot}} = -D\partial_x n(x,t) + \alpha n(x,t)$. Reflecting boundary conditions: individuals are confined in the region $0 \leq x \leq L$. Moreover, the advection term $\alpha n(x,t)$ indicates that individuals have a biased movement towards the right (wind? water flow?).

Example: cross section of a river (as in Exercise 17) with a water flux towards the right side of the river (maybe because the section is not perpendicular to the water flux).

(b) Let $\overline{n}(x)$ be an equilibrium solution. Then

$$0 = \frac{d}{dx}\left(D\overline{n}'(x) - \alpha\overline{n}(x)\right) \quad \Leftrightarrow \quad D\overline{n}'(x) - \alpha\overline{n}(x) = c \quad \text{for some } c \in \mathbb{R}$$

$$\Leftrightarrow \quad \overline{n}'(x) = \frac{\alpha}{D}\overline{n}(x) + \frac{c}{D} \quad \Leftrightarrow \quad \overline{n}(x) = e^{\frac{\alpha}{D}x}\left(\overline{n}(0) + \frac{c}{\alpha}\right) - \frac{c}{\alpha}$$

We now impose that $\overline{n}(x)$ satisfies the boundary conditions:

$$-\alpha e^{\frac{\alpha}{D}x}\overline{n}(0) - ce^{\frac{\alpha}{D}x} + \alpha e^{\frac{\alpha}{D}x}\overline{n}(0) + ce^{\frac{\alpha}{D}x} - c = 0 \quad \Leftrightarrow \quad c = 0.$$

Therefore, $\overline{n}(x) = e^{\frac{\alpha}{D}x}\overline{n}(0)$. Observe that the solution is well-defined up to a multiplicative constants (every multiple $k\overline{n}(x)$ with $k > 0$ is a solution of the equation). In order to fix a specific solution, we can choose an extra condition. For instance, you can choose $\overline{n}(0) = 1$, or normalize the total density to 1 ($1 = \int_0^L \overline{n}(x)dx$), or normalize the average population density to 1 ($1 = \frac{1}{L}\int_0^L \overline{n}(x)dx$).

(c) Main steps for local stability analysis:

1. Linearize the problem around the equilibrium $\overline{n}(x)$.

   This is done by considering a small perturbation $u(x,t)$ of the equilibrium $\overline{n}(x)$ and imposing that $n(x,t) = \overline{n}(x) + u(x,t)$ satisfies (2). By simplifying the equation using the equilibrium conditions, you get the linearized equation for $u(x,t)$. Remember that the linearized equation is given by a linear equation with zero boundary conditions.

2. Consider a trial solution of the linearized problem of the form $u(x,t) = v(x)e^{\lambda t}$, where $\lambda \in \mathbb{C}$ is an eigenvalue and $v(x)$ is the corresponding eigenfunction, and substitute this into the linearized problem. You will get the *characteristic equation* for λ and $v(x)$.

3. Solve the characteristic equation and determine the eigenvalues λ of the linearized equation. The equilibrium $\overline{n}(x)$ is stable if all the eigenvalues have negative real part; it is unstable if there exists at least one eigenvalue with positive real part.

Let us go back to our specific problem. Assume that, after normalization,

$$\overline{n}(x) = Ke^{\frac{\alpha}{D}x}$$

for a fixed $K > 0$, and remember the equilibrium conditions

$$\begin{cases} 0 = -\partial_x\left(-D\overline{n}'(x) + \alpha\overline{n}(x)\right) \\ 0 = -D\overline{n}'(x) + \alpha\overline{n}(x) \qquad \text{for } x = 0, x = L. \end{cases} \tag{3}$$

We now study the stability of the equilibrium. Consider $n(x,t) = \overline{n}(x) + u(x,t)$. The linearized problem (obtained by substituting $n(x,t)$ into (2) and by using the equilibrium conditions (3)) is

$$\begin{aligned} \partial_t u(x,t) &= -\partial_x\left(-D\partial_x u(x,t) + \alpha u(x,t)\right) \\ 0 &= -D\partial_x u(x,t) + \alpha u(x,t) \quad x = 0, L. \end{aligned} \tag{4}$$

(Note that, since the original problem is linear with zero boundary conditions, the linearized problem coincides with the original one.)

Consider a trial solution $u(x,t) = v(x)e^{\lambda t}$ for an eigenvalue $\lambda \in \mathbb{C}$ and the corresponding eigenfunction $v(x)$. By substituting into (4), we get

$$\begin{aligned} \lambda v(x)e^{\lambda t} &= Dv''(x)e^{\lambda t} - \alpha v'(x)e^{\lambda t} \\ 0 &= -Dv'(x)e^{\lambda t} + \alpha v(x)e^{\lambda t} \quad x = 0, L. \end{aligned}$$

By simplifying $e^{\lambda t}$, we conclude that the eigenvalue problem (*characteristic equation*) is

$$\begin{aligned} &Dv''(x) - \alpha v'(x) - \lambda v(x) = 0 \\ &0 = -Dv'(0) + \alpha v(0) \\ &0 = -Dv'(L) + \alpha v(L) \end{aligned}$$

The first equation is a second order linear ODE with constant coefficients. We solve it by the standard method that you can find in every textbook about ODEs. Consider the characteristic polynomial

$$Dy^2 - \alpha y - \lambda = 0$$

and denote $y_{1,2} = \frac{\alpha \mp \sqrt{\alpha^2 + 4\lambda D}}{2D}$ its distinct, real roots. Then, the solutions of the ODE are of the form

$$v(x) = Ae^{y_1 x} + Be^{y_2 x} \tag{5}$$

for $A, B \in \mathbb{R}$ (A, B not both equal to zero because $v(x)$ is an eigenfunction). We now impose the boundary conditions:

$$\begin{cases} v(x) = Ae^{y_1 x} + Be^{y_2 x} \\ D(Ay_1 + By_2) = \alpha(A + B) \\ D\left(Ay_1 e^{y_1 L} + By_2 e^{y_2 L}\right) = \alpha\left(Ae^{y_1 L} + Be^{y_2 L}\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} v(x) = Ae^{y_1 x} + Be^{y_2 x} \\ B = \frac{Dy_1 - \alpha}{-Dy_2 + \alpha} A = -A\frac{\alpha + \sqrt{\alpha^2 + 4\lambda D}}{\alpha - \sqrt{\alpha^2 + 4\lambda D}} \\ Ae^{y_1 L}(Dy_1 - \alpha) = Be^{y_2 L}(-Dy_2 + \alpha) \end{cases} \qquad \Leftrightarrow \begin{cases} v(x) = Ae^{y_1 x} + Be^{y_2 x} \\ B = \frac{Dy_1 - \alpha}{-Dy_2 + \alpha} A \\ e^{y_1 L} = e^{y_2 L} \end{cases}$$

Therefore, λ is an eigenvalue (i.e., it solves the characteristic equation) if and only if the following relation holds

$$y_2 L = y_1 L + 2k\pi i \quad \text{for some } k \in \mathbb{Z}.$$

Let us substitute the explicit values of y_1, y_2 and solve the condition for λ_k:

$$\begin{aligned} \frac{\alpha - \sqrt{\alpha^2 + 4\lambda_k D}}{2D} L &= \frac{\alpha + \sqrt{\alpha^2 + 4\lambda_k D}}{2D} L + 2k\pi i \\ \Leftrightarrow \quad \sqrt{\alpha^2 + 4\lambda_k D} &= -\frac{2Dk\pi i}{L} \\ \Leftrightarrow \quad \alpha^2 + 4\lambda_k D &= \left(\frac{2Dk\pi i}{L}\right)^2 \\ \Leftrightarrow \quad \lambda_k &= -D\frac{k^2\pi^2}{L^2} - \frac{\alpha^2}{4D}, \quad k \in \mathbb{Z} \end{aligned}$$

All eigenvalues λ_k, $k \in \mathbb{Z}$ are real and negative, therefore we conclude that the equilibrium $\overline{n}(x)$ is stable.