

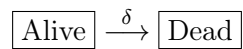
MATHEMATICAL MODELLING

HOMEWORK SOLUTIONS

September 9, 2015

Exercise 1

(a) Dying Poisson process: the only i -state transition is



Let $P(t)$ be the probability that an individual is still alive at time t . Then

$$\begin{aligned} P(t + \Delta t) &= \text{prob that } i \text{ was alive at time } t \text{ and has survived} \\ &= P(t)(1 - \delta\Delta t + O(\Delta t^2)) \end{aligned}$$

(it's true: higher order terms come from Taylor expansion of the Poisson distribution) and by rearranging and taking the limit $\Delta t \rightarrow 0$, we get

$$P'(t) = -\delta P(t).$$

This is a linear ODE and the solution is $P(t) = e^{-\delta t}P(0) = e^{-\delta t}$ (note that $P(0) = 1$ because we assume that the individual is alive at time 0).

(b) Consider the random variable T describing the lifetime of individual i . We derive the probability distribution

$$\begin{aligned} F_T(t) &= P(T \leq t) = P(i \text{ has already died at time } t) \\ &= 1 - P(i \text{ is alive at time } t) \\ &= 1 - e^{-\delta t}. \end{aligned}$$

The probability density is

$$f_T(t) = F_T'(t) = \delta e^{-\delta t}.$$

Verify that this is exactly the probability density of an exponentially distributed r.v. with mean δ^{-1} , or compute explicitly the mean value

$$\begin{aligned} E(T) &= \int_0^\infty t f_T(t) dt = \int_0^\infty t \delta e^{-\delta t} dt \\ &= \left[-te^{-\delta t} \right]_0^\infty + \int_0^\infty e^{-\delta t} dt = 0 + \frac{1}{\delta} \left[-e^{-\delta t} \right]_0^\infty = \frac{1}{\delta} \end{aligned}$$

where the first step is obtained through integration by parts.

(c) As explained in the lectures, through the law of large numbers (LLN) we can interpret the *individual probabilities* as *fractions of population* having a certain property. Define for each individual i the random variable

$$R_i(t) = \begin{cases} 1 & \text{if } i \text{ is alive at time } t \\ 0 & \text{otherwise} \end{cases}$$

Then, $E(R_i(t)) = 1P(t) + 0(1 - P(t)) = P(t)$, and by the law of large numbers,

$$\text{fraction of pop. in state } i = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n R_i(t) = P(t).$$

Denote x the initial population density, and

$$N(t) = P(t)x = \text{pop. density of alive individuals.}$$

Then, by differentiating, we get the ODE at the population level

$$N'(t) = P'(t)x = -\delta P(t)x = -\delta N(t).$$

Exercise 2

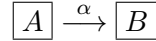
(a)	i-states \boxed{A} individuals in first habitat \boxed{B} individuals in second habitat	i-state transitions $\boxed{A} \xrightarrow{\alpha} \boxed{B}$ $\boxed{B} \xrightarrow{\beta} \boxed{A}$	differential equations $\frac{dA}{dt} = -\alpha A + \beta B$ $\frac{dB}{dt} = \alpha A - \beta B$
(b)	i-states \boxed{E} egg \boxed{J} juvenile	i-state transitions $\boxed{E} \xrightarrow{h} \boxed{J}$	differential equations $\frac{dE}{dt} = -hE$ $\frac{dJ}{dt} = hE$
(c)	i-states \boxed{A} individual	i-state transitions $\boxed{A} \xrightarrow{\beta} 2\boxed{A}$	differential equations $\frac{dA}{dt} = \beta A$
(d)	i-states \boxed{M} male \boxed{F} female \boxed{O} non-reproducing offspring	i-state transitions $\boxed{M} + \boxed{F} \xrightarrow{\beta} \boxed{O} + \boxed{M} + \boxed{F}$	differential equations $\frac{dO}{dt} = \beta MF$

(e)	i-states \boxed{P} predator \boxed{R} prey	i-state transitions $\boxed{P} + \boxed{R} \xrightarrow{\alpha} \boxed{P}$	differential equations $\frac{dR}{dt} = -\alpha PR$
(f)	i-states \boxed{I} single individual \boxed{P} pair of individuals	i-state transitions $\boxed{I} + \boxed{I} \xrightarrow{\alpha} \boxed{P}$ $\boxed{P} \xrightarrow{\beta} 2\boxed{I}$	differential equations $\frac{dI}{dt} = -2\alpha I^2 + 2\beta P$ $\frac{dP}{dt} = +\alpha I^2 - \beta P$
(g)	i-states \boxed{I} individual	i-state transitions $\boxed{I} + \boxed{I} \xrightarrow{\alpha} \boxed{I}$	differential equations $\frac{dI}{dt} = -\alpha I^2 + \frac{1}{2}\alpha I^2 = \frac{1}{2}\alpha I^2$

Density-independent: (a), (b), (c); density-dependent: (d), (e), (f), (g). Check if you can distinguish by yourself the cases for which the conservation law applies (and corresponding conserved quantities).

Exercise 3

Density-independent transition. Consider first the density-independent transition



(e.g., natural death, maturation of an individual, hatching of an egg, recovery from a disease, ...). Observe that, in the case the state \boxed{B} is “death”, we are exactly in the case of Exercise 1.

The ODE describing the evolution of the population density is

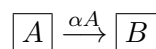
$$\begin{aligned} \frac{dA}{dt} &= -\alpha A \\ \frac{dB}{dt} &= +\alpha A. \end{aligned}$$

(observe that the total population size $N = A + B$ does not change). Therefore,

$$A(t) = e^{-\alpha t} A(0).$$

The expected time till a transition is obtained as in Exercise 1(b), so it is α^{-1} (check that you can prove it by yourself).

Density-dependent transition. Consider the density-dependent transition



(e.g., awakening, chemical reaction, migration/death caused by absence of space, ...)

The ODE for the population density is

$$\begin{aligned}\frac{dA}{dt} &= -\alpha A^2 \\ \frac{dB}{dt} &= +\alpha A^2.\end{aligned}$$

This is a nonlinear ODE. We solve it by separation of variables: by rearranging and integrating both sides we get

$$\int_{A(0)}^{A(t)} \frac{1}{A^2} dA = -\alpha t.$$

It is easy to see that $\int_{A(0)}^{A(t)} y^{-2} dy = A(0)^{-1} - A(t)^{-1}$ and therefore, by rearranging a little, we get the explicit solution

$$A(t) = \frac{A(0)}{1 + A(0)\alpha t}$$

Analogously as in Exercise 1, we denote T the random variable describing the time at which the transition happens. We compute the probability distribution

$$F_T(t) = P(T \leq t) = 1 - P(i \text{ is still in state } A \text{ at time } t) = 1 - A(t)$$

and the probability density function

$$f_T(t) = F'_T(t) = -A'(t)$$

Therefore, the expected value of T is (using integration by parts)

$$\begin{aligned}E(T) &= \int_0^\infty t f_T(t) dt \\ &= - \int_0^\infty t A'(t) dt = - [tA(t)]_0^\infty + \int_0^\infty A(t) dt \\ &= [\log(1 + A(0)\alpha t)]_0^\infty = +\infty\end{aligned}$$