# MATHEMATICAL MODELLING

HOMEWORK SOLUTIONS

September 9, 2015

## Exercise 1

**(a)** Dying Poisson process: the only $i$-state transaction is

$$\boxed{\text{Alive}} \xrightarrow{\delta} \boxed{\text{Dead}}$$

Let $P(t)$ be the probability that an individual is still alive at time $t$. Then

$$\begin{aligned} P(t + \Delta t) &= \text{prob that } i \text{ was alive at time } t \text{ and has survived} \\ &= P(t)(1 - \delta \Delta t + O(\Delta t^2)) \end{aligned}$$

(it's true: higher order terms come from Taylor expansion of the Poisson distribution) and by rearranging and taking the limit $\Delta t \to 0$, we get

$$P'(t) = -\delta P(t).$$

This is a linear ODE and the solution is $P(t) = e^{-\delta t} P(0) = e^{-\delta t}$ (note that $P(0) = 1$ because we assume that the individual is alive at time 0).

**(b)** Consider the random variable $T$ describing the lifetime of individual $i$. We derive the probability distribution

$$\begin{aligned} F_T(t) = P(T \le t) &= P(i \text{ has already dead at time } t) \\ &= 1 - P(i \text{ is alive at time } t) \\ &= 1 - e^{-\delta t}. \end{aligned}$$

The probability density is

$$f_T(t) = F_T'(t) = \delta e^{-\delta t}.$$

Verify that this is exactly the probability density of an exponentially distributed r.v. with mean $\delta^{-1}$, or compute explicitly the mean value

$$\begin{aligned} E(T) &= \int_0^\infty t f_T(t) dt = \int_0^\infty t \delta e^{-\delta t} dt \\ &= \left[ -t e^{-\delta t} \right]_0^\infty + \int_0^\infty e^{-\delta t} dt = 0 + \frac{1}{\delta} \left[ -e^{-\delta t} \right]_0^\infty = \frac{1}{\delta} \end{aligned}$$

where the first step is obtained through integration by parts.

**(c)** As explained in the lectures, through the law of large numbers (LLN) we can interpret the *individual probabilities* as *fractions of population* having a certain property. Define for each individual $i$ the random variable

$$R_i(t) = \begin{cases} 1 & \text{if } i \text{ is alive at time } t \\ 0 & \text{otherwise} \end{cases}$$

Then, $E(R_i(t)) = 1P(t) + 0(1 - P(t)) = P(t)$, and by the law of large numbers,

$$\text{fraction of pop. in state } i \ = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} R_i(t) = P(t).$$

Denote $x$ the initial population density, and

$$N(t) = P(t)x = \text{ pop. density of alive individuals.}$$

Then, by differentiating, we get the ODE at the population level

$$N'(t) = P'(t)x = -\delta P(t)x = -\delta N(t).$$

## Exercise 2

**(a)**

| i-states | i-state transitions | differential equations |
|---|---|---|
| $\boxed{A}$ individuals in first habitat<br>$\boxed{B}$ individuals in second habitat | $\boxed{A} \xrightarrow{\alpha} \boxed{B}$<br>$\boxed{B} \xrightarrow{\beta} \boxed{A}$ | $\frac{\mathrm{d}A}{\mathrm{d}t} = -\alpha A + \beta B$<br>$\frac{\mathrm{d}B}{\mathrm{d}t} = \alpha A - \beta B$ |

**(b)**

| i-states | i-state transitions | differential equations |
|---|---|---|
| $\boxed{E}$ egg<br>$\boxed{J}$ juvenile | $\boxed{E} \xrightarrow{h} \boxed{J}$ | $\frac{\mathrm{d}E}{\mathrm{d}t} = -hE$<br>$\frac{\mathrm{d}J}{\mathrm{d}t} = hE$ |

**(c)**

| i-states | i-state transitions | differential equations |
|---|---|---|
| $\boxed{A}$ individual | $\boxed{A} \xrightarrow{\beta} 2\boxed{A}$ | $\frac{\mathrm{d}A}{\mathrm{d}t} = \beta A$ |

**(d)**

| i-states | i-state transitions | differential equations |
|---|---|---|
| $\boxed{M}$ male<br>$\boxed{F}$ female<br>$\boxed{O}$ non-reproducing offspring | $\boxed{M} + \boxed{F} \xrightarrow{\beta} \boxed{O} + \boxed{M} + \boxed{F}$ | $\frac{\mathrm{d}O}{\mathrm{d}t} = \beta MF$ |

**(e)**

| i-states | i-state transitions | differential equations |
|---|---|---|
| $\boxed{P}$ predator<br>$\boxed{R}$ prey | $\boxed{P} + \boxed{R} \xrightarrow{\alpha} \boxed{P}$ | $\dfrac{\mathrm{d}R}{\mathrm{d}t} = -\alpha PR$ |

**(f)**

| i-states | i-state transitions | differential equations |
|---|---|---|
| $\boxed{I}$ single individual<br>$\boxed{P}$ pair of individuals | $\boxed{I} + \boxed{I} \xrightarrow{\alpha} \boxed{P}$<br>$\boxed{P} \xrightarrow{\beta} 2\boxed{I}$ | $\dfrac{\mathrm{d}I}{\mathrm{d}t} = -2\alpha I^2 + 2\beta P$<br>$\dfrac{\mathrm{d}P}{\mathrm{d}t} = +\alpha I^2 - \beta P$ |

**(g)**

| i-states | i-state transitions | differential equations |
|---|---|---|
| $\boxed{I}$ individual | $\boxed{I} + \boxed{I} \xrightarrow{\alpha} \boxed{I}$ | $\dfrac{\mathrm{d}I}{\mathrm{d}t} = -\alpha I^2 + \dfrac{1}{2}\alpha I^2 = \dfrac{1}{2}\alpha I^2$ |

Density-independent: (a), (b), (c); density-dependent: (d), (e), (f), (g). Check if you can distinguish by yourself the cases for which the conservation law applies (and corresponding conserved quantities).

## Exercise 3

**Density-independent transition.** Consider first the density-independent transition

$$\boxed{A} \xrightarrow{\alpha} \boxed{B}$$

(e.g., natural death, maturation of an individual, hatching of an egg, recovery from a disease,...). Observe that, in the case the state $\boxed{B}$ is “death”, we are exactly in the case of Exercise 1.

The ODE describing the evolution of the population density is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\alpha A$$
$$\frac{\mathrm{d}B}{\mathrm{d}t} = +\alpha A.$$

(observe that the total population size $N = A + B$ does not change). Therefore,

$$A(t) = e^{-\alpha t}A(0).$$

The expected time till a transition is obtained as in Exercise 1(b), so it is $\alpha^{-1}$ (check that you can prove it by yourself).

**Density-dependent transition.** Consider the density-dependent transition

$$\boxed{A} \xrightarrow{\alpha A} \boxed{B}$$

(e.g., awakening, chemical reaction, migration/death caused by absence of space,...)

The ODE for the population density is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\alpha A^2$$
$$\frac{\mathrm{d}B}{\mathrm{d}t} = +\alpha A^2.$$

This is a nonlinear ODE. We solve it by separation of variables: by rearranging and integrating both sides we get

$$\int_{A(0)}^{A(t)} \frac{1}{A^2}\,\mathrm{d}A = -\alpha t.$$

It is easy to see that $\int_{A(0)}^{A(t)} y^{-2}\,\mathrm{d}y = A(0)^{-1} - A(t)^{-1}$ and therefore, by rearranging a little, we get the explicit solution

$$A(t) = \frac{A(0)}{1 + A(0)\alpha t}$$

Analogously as in Exercise 1, we denote $T$ the random variable describing the time at which the transition happens. We compute the probability distribution

$$F_T(t) = P(T \leq t) = 1 - P(i \text{ is still in state } A \text{ at time } t) = 1 - A(t)$$

and the probability density function

$$f_T(t) = F_T'(t) = -A'(t)$$

Therefore, the expected value of $T$ is (using integration by parts)

$$\begin{aligned} E(T) &= \int_0^\infty t f_T(t) dt \\ &= -\int_0^\infty tA'(t)dt = -\left[tA(t)\right]_0^\infty + \int_0^\infty A(t)\,\mathrm{d}t \\ &= \left[\log(1 + A(0)\alpha t)\right]_0^\infty = +\infty \end{aligned}$$