MATHEMATICAL MODELLING
EXERCISE 6 – 7

6.

Consider the following variation on the site-competition model with i-states $\boxed{\text{S}}$ (free site), $\boxed{\text{X}}$ (occupied site; also: site owner) and $\boxed{\text{Y}}$ (individual searching for a free site), and individual-level processes modelled as

$$
\begin{array}{rcll}
\boxed{\text{X}} & \xrightarrow{\beta} & \boxed{\text{S}} + (n+1)\,\boxed{\text{Y}} & \text{(birth)} \\
\boxed{\text{X}} & \xrightarrow{\delta} & \boxed{\text{S}} & \text{(death)} \\
\boxed{\text{S}} + \boxed{\text{Y}} & \xrightarrow{\lambda} & \boxed{\text{X}} & \text{(colonization)} \\
\boxed{\text{Y}} & \xrightarrow{\mu} & \text{null} & \text{(death).}
\end{array}
$$

(a) give a more detailed interpretation of the process indicated as "birth";
(b) give the corresponding differential equations for the population densities of $\boxed{\text{S}}$, $\boxed{\text{X}}$ and $\boxed{\text{Y}}$;
(c) show that the total density of sites (i.e., occupied and free) stays constant, and use this to reduce the system to two equations: one for X and one for Y;
(d) turn the system into a fast-slow system by assuming that n and μ are large compared to all other parameters. To this end, use a small scaling parameter $\varepsilon > 0$ as we did in the lectures;
(e) show that in the limit for $\varepsilon \to 0$, the system becomes essentially one-dimensional as the dynamics becomes confined to a slow manifold. Give the equation for the slow manifold, and show that the slow dynamics on this manifold are given by the logistic equation.

7.

Consider the epidemic model

$$
\begin{cases}
\frac{ds}{dt} = -\beta si \qquad\qquad +\delta r & \text{(susceptible but healthy)} \\
\frac{di}{dt} = +\beta si \quad -\gamma i & \text{(infected)} \\
\frac{dr}{dt} = \qquad\qquad +\gamma i \quad -\delta r & \text{(recovered and temporally immune)}
\end{cases}
$$

Interpret the different terms on the right hand side in terms of individual level processes. Reduce the dimensionality of the system using a conservation relation. Do a phase-plane analysis and determine the local stability of all equilibria. You may have to consider different cases depending on the parameter values.