MATHEMATICAL MODELLING
EXERCISE 22 – 23

22.
Suppose the prey can be in one of two states, M_1 (freely roaming about) or M_2 (hiding for the predator), subject to the following transitions only:

$$\mathrm{M}_1 + \mathrm{P} \xrightarrow{\alpha} \mathrm{M}_2 + \mathrm{P}$$

$$\mathrm{M}_2 \xrightarrow{\tau^{-1}} \mathrm{M}_1$$

Suppose further that prey in state M_1 diffuse, but that prey in state M_2 sit still and thus do not diffuse. The predator, too, diffuses and in addition to that also has a positive taxis towards higher population densities of M_1, presumably because it does not see the hiding prey.

(a) Give the corresponding system of three (reaction-diffusion) equations for the corresponding population densities m_1, m_2 and p, subject to reflecting (i.e., zero-flux) boundaries. Simplify the expressions of the boundary conditions as much as possible.

(b) Assuming that the reactions (i.e., state transitions) are very fast compared to diffusion and taxis, calculate the quasi-equilibrium of the fast dynamics. Use this to reduce the number of equations from three to two: one for total prey density $m := m_1 + m_2$ and one for predator density p. Rewrite the boundary conditions accordingly.

(c) Rewrite the equations with a density-dependent diffusion coefficient $D_1(p)$ and density-dependent taxis coefficients $Q_1(p)$ and $Q_2(m, p)$, i.e., rewrite the equations in the form

$$\partial_t m = \partial_x \big(D_1(p)\partial_x m\big)$$

$$\partial_t p = D_2 \partial_{xx} p - \partial_x \big(Q_1(p)\, p\, \partial_x m\big) + \partial_x \big(Q_2(m, p)\, p\, \partial_x p\big)$$

Give explicit expressions for the density-dependent coefficients.

23.
Consider the predator-prey system

$$\varepsilon\,\partial_t m = a - b\,m - \beta m p$$

$$\partial_t p = \gamma\beta m p - \delta p + D\partial_{xx} p$$

Suppose that $0 < \varepsilon << 1$ so that the m-dynamics are much faster than the p-dynamics.

(a) Calculate the quasi-equilibrium of the m-dynamics, and use this to reduce the system to a single equation for p only.

(b) Investigate what travelling wave solutions are possible on the infinite domain if $a\gamma\beta > \delta b$.