

MATHEMATICAL MODELING
(example I)

—
This is an example based on an actual exam given in the past, but it has been adapted to the format of this year's exam.

You have four hours to answer the questions.

The use of lecture notes is NOT allowed.

QUESTION 1

Formulate a “SIS-model with predation” where the predator captures only infected prey, i.e., give the minimum set of i -states, then the i -level processes as monomolecular and bimolecular reactions, and then the population equations as a system of differential equations.

QUESTION 2

Consider the predator-prey model

$$\begin{cases} \frac{dn}{dt} = rn \left(1 - \frac{n}{K}\right) - \beta np \\ \frac{dp}{dt} = \gamma \beta np - \delta p - \varepsilon p^2 \end{cases}$$

with all positive parameters. Give an interpretation of the system in terms of i -level processes. Give a phase-plane analysis, find all non-negative equilibria and determine their stability, if necessary by using local stability analysis. Distinguish between the cases $\gamma\beta K \leq \delta$ and $\gamma\beta K > \delta$.

QUESTION 3

Consider the partial differential equation

$$\partial_t n = n(1 - n) - v\partial_x n + D\partial_{xx} n$$

with positive parameters v and D and boundary conditions $n(t, -\infty) = 1$ and $n(t, +\infty) = 0$. Interpret the equation in terms of i -level processes. Find a traveling wave solution $n(t, x) = \varphi(x - ct)$ for $c > v$. Give an expression for the minimum wave speed.

Continued on next page ...

QUESTION 4

Consider a vertical water column containing fish. Flying above the water are birds trying to catch fish that come to the surface. Formulate a reaction-diffusion model with appropriate boundary conditions for the population density of the fish and birds taking into account the following processes:

- The fish move randomly but also show negative auto-taxis, i.e., they are repelled by one another (presumably to avoid competition).
- Only fish that reach the surface may be picked up from the water by birds.
- The total density of birds density is fixed, but because handling the fish takes time (presumably because of having to land with the fish and eat it, before taking the wings again to catch a new fish) the population density of actually hunting birds depends on the fish density just below the water surface.