MATHEMATICAL MODELING
(example I)
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This is an example based on an actual exam given in the past, but it has been adapted to the format of this year's exam.
You have four hours to answer the questions.
The use of lecture notes is NOT allowed.

QUESTION 1

Formulate a population model which only includes the following three elements: (1) A predator searches for prey, but when it finds one, it will not immediately attack. Instead, it will start stalking the prey to get close without being seen. (2) If the prey discovers that it is being stalked, it immediately takes a defensive posture, upon which the predator gives up and goes away. (3) if the prey does not discover the stalking predator before the latter has come within striking range, the predator attacks and kills the prey.

Note: first define the i-states, then formulate the above processes as monomolecular or bimolecular reactions, and only then formulate the corresponding population equations.

QUESTION 2

Consider the predator-prey model

$$\begin{cases} \frac{dx}{dt} &= \alpha - x - \beta xy & \text{(prey)} \\ \\ \frac{dy}{dt} &= \gamma\beta xy - \delta y - \frac{\varepsilon y}{1+\varepsilon\tau y}H & \text{(predator)} \end{cases}$$

with all positive parameters. Interpret the system in terms of i-level processes. Give a phase-plane analysis, i.e., sketch the zero-clines, indicate the direction of the flows, indicate all non-negative equilibria and determine their local stability. Distinguish the two cases where the predator can invade the prey population and where the predator cannot.

Advice: It is not necessary nor advisable to actually calculate the equilibria explicitly.

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QUESTION 3

Consider the following system of partial differential equations:

$$\begin{cases} \partial_t U &= -\alpha U^2 + 2\beta V \\ \partial_t V &= \frac{1}{2}\alpha U^2 - \beta V + \mu\,\partial_x^2 V + \nu\,\partial_x\big(V\partial_x U\big) \end{cases}$$

with all positive parameters. Interpret the system in terms of i-level processes. Calculate all equilibria if both $x = 0$ and $x = 1$ are reflecting boundaries.

Hint: introduce the new variable W:=U+2V.

QUESTION 4

Suggest a set of differential equations plus appropriate boundary conditions for each of the following situations:

(a) A predator-prey system in which the predator moves towards higher prey densities and the prey towards lower predator densities. Assume further that the movement of the individual prey and the individual predator also has a random (i.e., undirected) component.

(b) A population of cells produces a substance that diffuses and is gradually broken down at some given rate. The cells move towards higher concentrations of the substance but also have a random component to their movement.

(c) Oxygen diffuses from the surface of a vertical water column to the bottom where it is absorbed by a layer of debris. The absorption rate is proportional to the density of the debris. The oxygen that is being absorbed is used (by bacteria) to decompose the debris, the density of which therefore decreases in time.