University of Helsinki / Department of Mathematics and Statistics
SCIENTIFIC COMPUTING
Exercise 08, 9.11.2015

N.B. The files mentioned in the exercises (if any) are available on the course homepage.

1. (a) Show that for a square matrix a, if λ is an eigenvalue of a (i.e. for some $x \neq 0, ax = \lambda x$), then $\lambda + c$ is an eigenvalue of $a + c * I$ where I is the identity matrix of the same size as a.

(b) Let a_1 be a random $n \times n$ matrix, $a = a_1 + a_1^T$, and define t to be the least eigenvalue of a if it is < 0 and $= 0$ otherwise. Show that $a - tI$ has nonnegative eigenvalues and that it is symmetric.

(c) Generate with (b) symmetric square matrices a with nonnegative eigenvalues and show that if $[u, s, v] = svd(a)$ and $ss = sqrt(s), w = u * ss$, then $a = w * w^T$.

2. The vertices of a quadrilateral are $(0,0), (p_1, p_2), (1, 0), (q_1, q_2)$, where $p_2 < 0, q_2 > 0$. Generate such quadrilaterals and compute their areas. Show that Bretschneider's formula for the area holds:

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \alpha}$$

where α is the mean value of the angles at p and q and a, b, c, d are the sides, $2s = a + b + c + d$. Hint: Use polyarea to check the formula.

3. Let B and C be $m \times n$ matrices with real entries and let $A = B + iC$. Relate the singular values of A to those of

$$\begin{bmatrix} B & -C \\ C & B \end{bmatrix}.$$

Observe this gives an idea to reduce the computation of the SVD of complex $m \times n$ matrices to the case of real $2m \times 2n$ matrices.

4. Recall from linear algebra that if $\Delta \equiv ad - bc \neq 0$, then

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

FILE: ~/MME/demo13/d08/d08.tex — 1. marraskuuta 2015 (klo 16.48).

Consider the Newton method $x_{n+1} = x_n - J_f(x_n)^{-1} f(x_n)$ to solve the system

$$\begin{cases} x_1^2 + x_2^2 - 1 & = 0 \\ x_1^2 - x_2^2 & = 0 \end{cases}$$

with the initial point $(0.5, 0.5)$. Use standard methods from linear algebra and multidimensional calculus, such as the above formula for matrix inverse, to compute the following expressions analytically and to print the numerical values $J_f(x_n)$, $J_f(x_n)^{-1}, T(x_n) \equiv J_f(x_n)^{-1} f(x_n)$ for the first three iteration steps.

5. For $t \in R \setminus \{0\}$, the pseudo-inverse is defined by $t^+ = 1/t$ and we set $0^+ = 0$. If s is a $m \times n$ diagonal matrix let s^+ be the $n \times m$ diagonal matrix obtained by applying this operation elementwise: $(s^+)_{i,j} = s_{i,j}^+$. For an $m \times n$ matrix a with a SVD $a = usv^T$ we define $a^+ = vs^+u^T$. Check for a number of random matrices whether and how well a^+ agrees with the MATLAB pseudo-inverse $pinv(a)$.

6. Finding the root of a linear $n \times n$ system $ax = b$ can be considered as a minimization problem for the function $g(x) = |a*x - b| = norm(a*x - b)$. Use this idea to solve the problem $a * x = b$ when a is $n \times n$ Hilbert matrix. Report for each $n = 2:9$ the residual $|ax - b|$ and $|xnum - exact|$ when exact= ones(n.1) and b=a*exact.