

University of Helsinki / Department of Mathematics and Statistics
SCIENTIFIC COMPUTING
Exercise 04, 5.10.2015

N.B. The files mentioned in the exercises (if any) are available on the course homepage

1. (a) Let X and Y be independent uniformly distributed random variables on $(0, 1)$. As we know, samples of X can be generated by `x= rand(1, 100)`; for instance. Now it is a basic fact (this need not be proven) that the new random variables

$$U = \cos(2\pi X)\sqrt{-2 \log Y}; \quad V = \sin(2\pi X)\sqrt{-2 \log Y}$$

follow the normal distribution with parameters $(0, 1)$, i.e. with mean 0 and variance 1. Use this so called Box-Müller method to generate 200 samples of normal distribution, plot the result with the command `hist`, compute the mean and standard deviation of the sample.

(b) The amplitude distribution of a signal sent by a mobile phone to a base station follows so called Rayleigh distribution. Suppose that X_1, X_2 are zero-mean normally distributed random variables with variance σ^2 and define a new random variable R by $R = \sqrt{X_1^2 + X_2^2}$. Then R follows the Rayleigh distribution. Generate 100 samples of a Rayleigh distribution and plot the histogram.

2. Suppose that $f: [a, b] \rightarrow [0, \infty)$ is continuous and that $0 \leq f(x) \leq M$ for all $x \in [a, b]$. Use the Monte Carlo method to approximate the value of

$$\int_a^b f(x) dx,$$

that is, choose m random points in $[a, b] \times [0, M]$ and compute the ratio p/m where p is the number of points below the graph of $f(x)$. Apply this method for the function

$$f(x) = \sum_{j=1}^n c_j (1 + \sin(d_j x))$$

in $[0, 1]$ with $m = 10j$, $j = 10 : 10 : 100$ where $n = 4$, $c = \text{rand}(1, n)$, $d = 1 + 3 * \text{rand}(1, n)$. Compare your result to the exact value

$$\int_a^b f(x) dx = (b - a) \sum_{j=1}^n c_j + \sum_{j=1}^n (c_j / d_j) (\cos(d_j * a) - \cos(d_j * b)),$$

see Problem 3/Exercise 2.

3. The ASCII codes of capital letters A,...,Z are 65,...,90. A simple ciphering method, so called Caesar cipher, is the following. Fix an integer $p \in [1, 25]$. Each letter is replaced by another, obtained by increasing its ASCII code by the constant p . (Note that we recycle: 91 corresponds to 65 i.e. after Z come A,B,C,...). The program hlp043.m shows how this happens. Use this idea to decipher the message:

Q C A D I H C S F U C G I A

4. We want to fit a model of the form $f(x) = ae^{bx}$ to the data set

x	1	3	4	6	9	15
y	4.0	3.5	2.9	2.5	2.75	2.0

where a and b are parameters to be determined from the data.

(a) For this purpose we introduce new transformed variables $X = x$, $Y = \log(y)$. Carry out this data transformation and print out the transformed variables.

(b) After the transformation the new model is $F(x) = \log f(x) = bx + \log a$. Apply the usual LSQ method to find b and $\log a$.

(c) Print the results in the following format

x(i)	y(i)	Y(i)	a*exp(b*x(i))	y(i)-a*exp(b*x(i))
1	4.0		
.....				
15	2.0			

and plot the data and the fitted curve in the same figure.

5. For a complex $n \times n$ matrix a let $P_i = \sum_{j=1, j \neq i}^n |a_{i,j}|$, $m_0 = \min\{|a_{i,i}| - P_i : i = 1, \dots, n\}$, $m = \max\{m_0, 0\}$, $M = \max\{|a_{i,i}| + P_i : i = 1, \dots, n\}$.

By Gerschgorin's theorem (recall Exercise 03) the eigenvalues λ_i of a satisfy

$$m \leq |\lambda_i| \leq M; \quad i = 1, \dots, n$$

and it also follows that $m^n \leq D \leq M^n$, $D = |\det(a)|$. Set $m_1 = \min\{|\lambda_i| : i = 1, \dots, n\}$ and $m_2 = \max\{|\lambda_i| : i = 1, \dots, n\}$. Write a MATLAB script that experimentally confirms these statements, by printing out the test results in the following format

```
n    m    m1    m2    M    D - m^n    M^n -D
```

Use random complex $n \times n$ matrices, $n=5:5:50$.

Repeat the experiment for the matrices $a=2*n*eye(n)+rand(n,n)+i*rand(n,n)$.

6. The arithmetic-geometric mean $ag(a, b)$ of two positive numbers $a > b > 0$ is defined as $ag(a, b) = \lim a_n$, where $a_0 = a$, $b_0 = b$, and

$$a_{n+1} = (a_n + b_n)/2, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n = 0, 1, 2, \dots$$

(a) Write a function, which takes two arguments (double), computes ag and returns the value (double).

(b) The hypergeometric function ${}_2F_1(a, b; c; x)$ is defined as a sum of the series,

$$\begin{aligned} {}_2F_1(a, b; c; x) = & 1 + \frac{abx}{c \cdot 1!} + \frac{a(a+1)b(b+1)x^2}{c(c+1) \cdot 2!} + \dots \\ & + \frac{a(a+1)\dots(a+j-1)b(b+1)\dots(b+j-1)x^j}{c(c+1)\dots(c+j-1) \cdot j!} + \dots \end{aligned}$$

This hypergeometric series converges for $\text{abs } x < 1$. Gauss proved in 1799 that there is a connection between the hypergeometric function and the arithmetic-geometric mean,

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \frac{1}{ag(1, \sqrt{1-r^2})}$$

for $0 < r < 1$. Tabulate the difference of the two sides of this identity for $r = 0.05k$, $k = 1, \dots, 19$. Use the routine on the web-page to calculate the values of the ${}_2F_1$ or the MATLAB built-in program (`help hypergeom`).