Logic Toolbox
Department of Mathematics and Statistics, University of Helsinki
Fall 2015
Exercise 6

1. Prove the omitted atomic case of Łos's theorem (formulas of the form $R(t_1, \ldots, t_n)$).

2. Prove the connective and universal quantifier steps of the induction in Łos's theorem.

3. Prove that finiteness is not axiomatizable, i.e. there is no set of sentences T such that a structure $\mathcal{M}$ is a model of T if and only if $\mathcal{M}$ is finite.

4. Prove that if φ is a sentence of the language of fields which is true in every field of characteristic zero, then there is a finite set of primes P_φ such that φ is true in every field of characteristic p if $p \notin P_\varphi$.

5. Let K_1 and K_2 be two classes of models. Let T_l be the theory of K_l (the set of all sentences which hold in every model in K_l) for $k = 1, 2$. Prove that $T_1 \cup T_2$ is satisfiable if and only if some ultraproduct of members of K_1 is elementarily equivalent to (i.e. satisfies exactly the same sentences as) some ultraproduct of members of K_2.

6. Show that the ordering of the real numbers is not isomorphic to an ultrapower of the ordering of the rationals. Hint: An ultrapower of $\mathbb{Q}$ is not complete.