Logic Toolbox
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Exercise 5

1. Show that if $|A_i| \leq |B_i|$ for all $i \in I$, then $|\prod_{i \in I} A_i/D| \leq |\prod_{i \in I} B_i/D|$.

An ultrafilter D over a set I is λ-*complete*, if for every set F of fewer than λ sets in D, the intersection $\bigcap F$ is also in D. Every ultrafilter is ω-complete but the existence of non-principal $\aleph_1$-complete ultrafilters is not provable from the ZFC axioms.

2. Show that if an ultrafilter D on I is not $\aleph_1$-complete, then every ultraproduct $\prod_{i \in I} A_i/D$ has cardinality $< \omega$ or $\geq 2^\omega$.

3. Show that if B is an ultraproduct of finite structures then $|B|$ is either finite or $\geq 2^\omega$.

4. Show that the following conditions on an ultrafilter D over I are equivalent:

(1) D is not $\aleph_1$-complete.
(2) There are disjoint non-empty sets $X_i \subseteq I$, $i < \omega$, such that for each $n < \omega$, $\bigcup_{i \geq n} X_i \in D$.
(3) The ultrapower $(\omega, <)^I/D$ is not well-ordered.

5. Let E be a countable subset of $\mathcal{P}(\mathbb{N})$. Show that the filter generated by E cannot be a non-principal ultrafilter.

6. Show that for any ultrafilter D over I, D is non-principal if and only if D contains the cofinite filter on I.