Introduction

The area of mathematical logic is full of useful methods that often remain unknown to a wider audience. This course aims at bringing forth some of the most basic (and probably most useful) ones. The course is mainly aimed at non-logicians, but may have something to offer also for a beginning logic student.

1 Ordinals and cardinals

The first tool from logic (actually set theory) is the possibility to count with infinities and do constructions indexed past the natural numbers. The key to do this is to find a definition of number that allows for generalization. Here one needs to note that there are two aspects to being a number. One is assigning an order: we can talk about the first, second, third, etc. element in a sequence. The other aspect is that of quantity: how many elements are there in a given set? So there are two different equivalence relations at play. One is that of being order-isomorphic, the other of being equinumerous. As long as one only looks at finite orders and sets, these give the same concept of numberb but things look different when once one gets past the finite.

The development of set theory is closely tied to the search for a solid foundation of mathematics. This course does not go into the question of how mathematics can be rigorously built on set theory. When formal properties of sets are referred to, we tacitly assume that we are working in Zermelo-Fraenkel set theory with Axiom of Choice (ZFC). A list of the axioms of ZFC can be found in the appendix.

1.1 Ordinals

Ordinals generalize the idea of numbers as ordering things (*first*, *second*,...) The method of doing this is to define ordinals so that they represent types of well-orderings. Recall that a linear order is a binary relation $<$ that is

- irreflexive ($\forall x x \not< x$),
- transitive ($\forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x < z)$) and
- connected ($\forall x \forall y(x < y \vee y < x \vee x = y)$).

Definition 1.1. A *well ordering* on A is a linear ordering on A with the further property that every nonempty subset of A has a least element.

Example 1.2. Examples and non-examples

- $(\mathbb{N}, <)$ is a well-order.
- $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$ are not well-orders.
- $(\mathbb{N}, <) + (\mathbb{N}, <)$ is a well order (two copies of $\mathbb{N}$ one after the other)

Definition 1.3. A set x is called *transitive* if and only if $\bigcup x \subseteq x$, i.e.,

$$z \in y \in x \Rightarrow z \in x.$$

Definition 1.4. An *ordinal number* is a transitive set well-ordered by $\in$.

This definition of an ordinal sounds rather artificial, but it has all the properties we need and makes notation nice, so it has become standard. Now, of course the first question one should check is that there actually are ordinals:

Example 1.5. 1. $\emptyset$ is an ordinal.

2. If α is an ordinal, then its *successor*

$$\alpha \cup \{\alpha\}$$

is an ordinal. (This is in fact the least ordinal greater than α, which we will see later.)

Proof. 1. Trivial.

2. For transitivity, let $x \in y \in \alpha \cup \{\alpha\}$. Now if $y \in \alpha$ we can use the fact that α is transitive, so $x \in \alpha \subseteq \alpha \cup \{\alpha\}$. Otherwise $y \in \{\alpha\}$, i.e. $y = \alpha$. But then $x \in \alpha \subseteq \alpha \cup \{\alpha\}$ by assumption.

   We need to show that $\alpha \cup \{\alpha\}$ is a well order. It is easy to show that $\alpha \cup \{\alpha\}$ is a linear order (α+one new largest point). So let $A \subset \alpha \cup \{\alpha\}$ be nonempty. If A is the singleton set $\{\alpha\}$, then α is a least element. Otherwise $A \backslash \{\alpha\}$ is nonempty and a subset of α, so it has a least element a in α. But then $a \in \alpha \cup \{\alpha\}$ and is least, since α is greater than all elements of α (in the $\in$-order).

□

This way we get the von Neumann natural numbers:

$$\begin{aligned}
&0 := \emptyset, \\
&1 := 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}, \\
&2 := 1 \cup \{1\} = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \\
&3 := 2 \cup \{2\} = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\vdots
\end{aligned}$$

Now we start gathering some properties of ordinals.

Lemma 1.6. *1. Every element of an ordinal is an ordinal.*

2. If α, β are ordinals, then $\alpha \subseteq \beta$ if and only if $\alpha \in \beta$ or $\alpha = \beta$.

3. Every set of ordinals is well ordered by $\in$.

Proof. 1. Let α be an ordinal and $x \in \alpha$. Now $x \subseteq \alpha$ (as α is transitive) so it inherits the well-ordering from α. So let $z \in y \in x$. Then $z \in y \in \alpha$, so $z \in \alpha$. Thus x, y, z are elements of α, which is linearly ordered by $\in$, in particular transitive. So $z \in y \in x$ implies $z \in x$, proving x transitive.

2. Exercise.

3. For irreflexivity, note that no ordinal α is a member of itself, since it would contradict the well-ordering of α. Transitivity follows from the transitivity of ordinals. For connectivity, let α and β be ordinals. Let $\gamma = \alpha \cap \beta$. If $\gamma = \alpha$ or $\gamma = \beta$ we are done by (2). Otherwise, since $\gamma \subseteq \alpha$ and by (2), $\gamma \in \alpha$. Similarly $\gamma \in \beta$. So $\gamma \in \alpha \cap \beta = \gamma$, which is a contradiction. Finally, let A be a nonempty set of ordinals. Let $x = \bigcap A$. Now $x \subseteq \alpha$ for all $\alpha \in A$, so by (2) and (1) it is an ordinal and by (2) it is a lower bound.

□

Now we have seen that any set of ordinals provides a well ordering, ordered by $\in$ and we adopt the convention of writing $\alpha < \beta$ for $\alpha \in \beta$ and $\alpha \leq \beta$ for $\alpha \in \beta$ or $\alpha = \beta$.

Note that by lemma 1.6 each ordinal α is equal to the set of all ordinals $\beta < \alpha$ and $\alpha \subseteq \beta$ if and only if $\alpha \leq \beta$.

The *infimum* of a nonempty set S of ordinals is the least element of S, and the *supremum* of S is the least ordinal which is greater than or equal to every element of S.

Corollary 1.7. 1. *Any transitive set of ordinals is itself an ordinal.*

2. *If S is a nonempty set of ordinals, then $\bigcup S$ is an ordinal.*

Remark 1.8. Note that $\bigcap S = \inf S$ and $\bigcup S = \sup S$. If S contains a greatest element β then $\sup S = \beta$. Otherwise $\sup S > \beta$ for all $\beta \in S$.

Definition 1.9. We denote the successor of an ordinal α by $\alpha + 1$, i.e.,

$$\alpha + 1 = \alpha \cup \{\alpha\}.$$

An ordinal α is called a *successor ordinal* if $\alpha = \beta + 1$ for some ordinal β. Otherwise it is called a *limit ordinal*.

Example 1.10. 1. $1 = 0 + 1 = \{\emptyset\}$ is a successor ordinal.

2. 0 is a limit ordinal.

3. $\omega = \sup \mathbb{N} = \{0, 1, 2, \dots\}$ is a limit ordinal.

4. $\omega + 1 = \omega \cup \{\omega\}$ is a successor ordinal.

The collection of all ordinals is denoted by Ord. It is too big to be a set, so it is a *proper class*. This is stated in the sc. Burali-Forti Paradox: If Ord were a set, it would be a transitive well-ordered set, i.e., an ordinal, but then Ord $\in$ Ord, which is not allowed by the ZF axioms. (see e.g. [End77])