Introduction to Number Theory
7. exercise set, solutions

1. Note that $x^2 = 7x + 1$ is equivalent to $x = 7 + 1/x$. Iterating this we get

$$x = 7 + \frac{1}{x} = 7 + \frac{1}{7 + \frac{1}{x}} = 7 + \frac{1}{7 + \frac{1}{7+\frac{1}{x}}} = \cdots$$

so the continued fraction representation of the positive root is $\{7; \overline{7}\}$. Easy computer check gives that with 6 partial denominators one obtains the accuracy in 10 decimals.

2. We use induction on n. Statement is clear for $n = 1$. Assume that it holds for some n. Then, by using recursion formulas proved in the lectures, we get

$$\begin{aligned} q_n p_{n+1} - q_{n+1} p_n &= q_n(\lambda_n p_n + p_{n-1}) - (\lambda_n q_n + q_{n-1}) p_n \\ &= \lambda_n (q_n p_n - q_n p_n) + q_n p_{n-1} - q_n p_{n-1} - q_{n-1} p_n \\ &= -(q_{n-1} p_n - q_n p_{n-1}) \\ &= -(-1)^{n-1} = (-1)^n, \end{aligned}$$

as desired. □

3. (i) Write $\alpha = m/n$. Let p/q be a rational number s.t.

$$\left| \frac{m}{n} - \frac{p}{q} \right| \leq \frac{c}{q^2}. \tag{1}$$

Then $|q(mq - np)| \leq c|n|$. Then if $m/n \neq p/q$ we only have finitely many choices for q. It is clear that for a fixed q the inequality (1) is violated when $|p|$ is large enough. Thus there are only finitely many rationals p/q s.t. (1) holds. □

(ii) Let α be the value of simple infinite continued fraction. Since $|\alpha - p_n/q_n| \leq q_n^{-2}$ holds for any $n \geq 1$ and convergents are reduced fractions, there are infinitely many rationals p/q s.t. the inequality $|\alpha - p/q| \leq q^{-2}$ holds. If α would be rational, this contradicts to part (i). □

4. We have calculated that the continued fraction representation of the golden ration is $\{1; \overline{1}\}$ in the previous exercises. To prove that the n^{th} convergent is F_{n+2}/F_{n+1} we use induction on n. The statement is clear for $n = 1$. If $p_n/q_n = F_{n+2}/F_{n+1}$, then

$$\frac{p_{n+1}}{q_{n+1}} = 1 + \frac{1}{p_n/q_n} = 1 + \frac{F_{n+1}}{F_{n+2}} = \frac{F_{n+1} + F_{n+2}}{F_{n+2}} = \frac{F_{n+3}}{F_{n+2}},$$

as desired. □

Heuristically the golden ration is the most difficult one to approximate by rationals as it's continued fraction consists of only ones. Namely, then it's convergents have small denominators, which are not that good approximations.

5. The continued fractions can be calculated by using Theorem 5.11. The answers are

(i) $\sqrt{11} = \{3; \overline{3, 6}\}$.

(ii) $\sqrt{13} = \{3; \overline{1, 1, 1, 1, 6}\}$.

6. (i) By using the answer of the previous problem we can calculate convergents (p_n, q_n). We know from the lectures that the fundamental solution for $x^2 - Dy^2 = 1$ will be (p_k, q_k)

with k as small as possible. By calculating values of $p_n^2 - 11q_n^2$ until we get 1, we find that the fundamental solution is $(10, 3)$.

(ii) This is similar as the previous part but now the expression to consider is $p_n^2 - 13q_n^2$. One finds that the fundamental solution is $(649, 180)$.

7. Assume that the sequence $\{\lambda_n\}$ is uniformly bounded. Then there exists $M > 0$ s.t. $\lambda_n \leq M$ for all $n \in \mathbb{N}$. Now by Corollary 5.12. we have

$$\left|x - \frac{p_n}{q_n}\right| > \frac{1}{(q_{n+1}+q_n)q_n} = \frac{1}{(\lambda_{n+1}q_n + q_{n-1} + q_n)q_n} \geq \frac{1}{\lambda_{n+1}q_n^2 + 2q_n^2} \geq \frac{1}{(M+2)q_n^2}.$$

On the other hand, by Theorem 5.16. if

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then p/q is convergent. Therefore we have for all rational p/q that

$$\left|x - \frac{p}{q}\right| > \min\left\{\frac{1}{2}, \frac{1}{M+2}\right\} \cdot \frac{1}{q^2} = \frac{1}{(M+2)q^2}.$$

Assume that there is a constant $c > 0$ s.t. for all $p/q \in \mathbb{Q}$ the inequality

$$\left|x - \frac{p}{q}\right| \geq \frac{c}{q^2}$$

holds. Then by Corollary 5.12. we have

$$\frac{c}{q_n^2} \leq \left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

for all $n \geq 0$. This gives $q_n > cq_{n+1} = c\lambda_{n+1}q_n + cq_{n-1}$ which yields

$$\lambda_{n+1} < \frac{1}{c} - \frac{q_{n-1}}{q_n} < \frac{1}{c},$$

so the sequence $\{\lambda_n\}$ is uniformly bounded. □

8*. (i) We proceed by induction on n. For $n = 0, 1$ the statement is clear. Assume that the statement holds for some n. Then we have

$$q_{n+1} \leq \lambda_{n+1}q_n + q_{n-1} \leq \lambda_{n+1}2^n \prod_{i=1}^{n} \lambda_i + 2^{n-1} \prod_{i=1}^{n-1} \lambda_i = 2^{n-1}(2\lambda_{n+1}\lambda_n + 1) \prod_{i=1}^{n-1} \lambda_i \leq 2^{n+1} \prod_{i=1}^{n+1} \lambda_i,$$

as desired. □

(ii) By part (i) we have $q_n \leq 2^{n+1!+2!+\ldots+n!} \leq 2^{3\cdot n!}$ for every $n \in \mathbb{N}$, where the last inequality follows from an easy induction. For the sake of contradiction, assume that β is algebraic of degree $\ell \geq 2$ (if β is rational, the statement is clear). Then by Liouville's theorem there exists $c > 0$ s.t. the inequality

$$\left|\beta - \frac{p_n}{q_n}\right| \geq \frac{c}{q_n^\ell} \geq \frac{c}{q_n^2 q_n^{\ell-2}} \geq \frac{c}{q_n^2 2^{3(\ell-2)\cdot n!}}$$

holds for every $n \in \mathbb{N}$. On the otherhand,

$$\left|\beta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2 \lambda_{n+1}} = \frac{1}{q_n^2 2^{(n+1)!}}$$

for every $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, we have

$$\frac{1}{q_n^2 2^{(n+1)!}} > \frac{c}{q_n^2 2^{3(\ell-2)\cdot n!}}.$$

This, however, cannot clearly hold when n is large enough. This contradiction shows that β is transcendental. $\square$