Introduction to Number Theory
6. exercise set, solutions

1. Note that $x^2 - Dy^2 = 1$ is equivalent to $1 + Dy^2 = x^2$. For a given D we find the minimal solution by going through values of y starting from 1 in $1 + Dy^2$ until we get a square. The following table shows the minimal solution for each D:

$$\begin{aligned} D &= 2 : (3, 2) \\ D &= 3 : (2, 1) \\ D &= 5 : (9, 4) \\ D &= 6 : (5, 2) \\ D &= 7 : (8, 3) \\ D &= 8 : (3, 1) \\ D &= 10 : (19, 6) \end{aligned}$$

2. First note that if $\left|\frac{p}{q} - \sqrt{3}\right| \geq 1/10$, then the statement is clearly true. Hence we assume that $\left|\frac{p}{q} - \sqrt{3}\right| < 1/10$. Now we simply estimate

$$\left|\frac{p}{q} - \sqrt{3}\right| = \frac{\left|\frac{p^2}{q^2} - \sqrt{3}\right|}{\left|\frac{p}{q} + \sqrt{3}\right|} = \frac{\left|p^2 - 3q^2\right|}{q^2 \cdot \left|\frac{p}{q} + \sqrt{3}\right|} \geq \frac{1}{q^2 \cdot \left|\frac{p}{q} + \sqrt{3}\right|} \geq \frac{1}{q^2 \cdot (\frac{1}{10} + 2\sqrt{3})} > \frac{1}{10q^2},$$

where we used the fact that $\sqrt{3} \notin \mathbb{Q}$ and the triangle inequality. □

3. Set $x = \sqrt{2} - \sqrt[3]{3}$. Then $3 = (\sqrt{2} - x)^3$ which gives $x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} + 3 = 0$ or $x^3 + 6x + 3 = \sqrt{2}(3x^2 + 2)$. Squaring leads to $x^6 - 6x^4 + 6x^3 + 12x^2 + 36x + 1 = 0$. Conversely, it is easy to check that $x = \sqrt{2} - \sqrt[3]{3}$ is indeed a root of this polynomial. As the polynomial has integer coefficients, it follows that $x = \sqrt{2} - \sqrt[3]{3}$ is algebraic. □

4. (a) Simply note that

$$\frac{57}{111} = \frac{1}{\frac{111}{57}} = \frac{1}{1 + \frac{54}{57}} = \frac{1}{1 + \frac{1}{\frac{57}{54}}} = \frac{1}{1 + \frac{1}{1 + \frac{3}{54}}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{18}}}.$$

Thus $57/111 = \{0; 1, 1, 18\}$.

Parts (b)-(d) can be done using the algorithm of Theorem 5.11.. The answers are

(b) $\sqrt{3} = \{1; 1, 2, 1, 2, ...\} = \{1; \overline{1, 2}\}$.

(c) $\frac{1+\sqrt{5}}{2} = \{1; 1, 1, 1, 1, ...\} = \{1; \overline{1}\}$.

(d) $e = \{2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...\}$.

5. Let us prove that every number of the form $\frac{a+\sqrt{D}}{b}$, where $a, b, D \in \mathbb{Z}, b \neq 0$ and $\sqrt{D} \notin \mathbb{N}$, are algebraic. Denote $x = \frac{a+\sqrt{D}}{b}$. Then $D = (bx - a)^2$ so $b^2x^2 - 2abx + (a^2 - D) = 0$ which proves the claim.

We still have to show that every second degree algebraic is of the required form. Let x be a second degree algebraic number. Then it satisfies a polynomial equation $ax^2 + bx + c = 0$ with integer coefficients. By solving this we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac$ is a square, then x would be rational and hence degree one algebraic number. Thus x has the required form. $\square$

6. (a) We know that $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$ for all $n \geq 1$. Thus we get that

$$\begin{aligned} x_{k+1} + y_{k+1}\sqrt{D} &= (x_1 + y_1\sqrt{D})^{k+1} \\ &= (x_1 + y_1\sqrt{D})^k(x_1 + y_1\sqrt{D}) \\ &= (x_k + y_k\sqrt{D})(x_1 + y_1\sqrt{D}) \\ &= x_k x_1 + y_1 y_k D + (x_1 y_k + x_k y_1)\sqrt{D} \end{aligned}$$

from where we conclude that

$$\begin{cases} x_{k+1} = x_1 x_k + y_1 y_k D \\ y_{k+1} = y_1 x_k + x_1 y_k \end{cases}$$

(b) We calculate

$$\begin{aligned} x_{k+1} &= x_1 x_k + y_1 y_k D \\ &= x_1 x_k + y_1(y_1 x_{k-1} + x_1 y_{k-1})D \\ &= x_1 x_k + y_1^2 D x_{k-1} + y_1 x_1 y_{k-1} D \\ &= x_1 x_k + (x_1^2 - 1)x_{k-1} + y_1 x_1 y_{k-1} D \\ &= x_1 x_k + x_1(x_1 x_{k-1} + y_1 y_{k-1} D) - x_{k-1} \\ &= x_1 x_k + x_1 x_k - x_{k-1} \\ &= 2x_1 x_k - x_{k-1}, \end{aligned}$$

as desired.

Basically an identical calculation shows that

$$y_{k+1} = 2x_1 y_k - y_{k-1}.$$

7. Write $q_n = 2^{(n-1)!}$ and $p_n = q_n(1 \pm 2^{-1!} \pm 2^{-2!} \pm \cdots \pm 2^{-(n-1)!})$ for every $n \in \mathbb{N}$. Then by Corollary 5.6. we have

$$\left|\xi - \frac{p_n}{q_n}\right| \leq \frac{2}{q_n^n}$$

for all $n \in \mathbb{N}$. Suppose that ξ is rational, $\xi = a/b$. As we clearly have $\xi \neq \xi_n$ for every $n \in \mathbb{N}$, it follows that

$$\left|\xi - \frac{p_n}{q_n}\right| \geq \frac{1}{bq_n}.$$

In particular,

$$\frac{2}{q_n^n} \geq \frac{1}{bq_n},$$

for every $n \in \mathbb{N}$. However, this obviously cannot hold when n is large enough. Therefore ξ is irrational for any choice of signs. $\square$

8*. Let D be a positive integer, which is not a square. Let c be a constant s.t. the inequality

$$\left|\frac{p}{q} - \sqrt{D}\right| \geq \frac{c}{q^2}$$

holds for all rational numbers p/q. By using arguments of problem 2. one sees that we can take $c = 1/(4\sqrt{D})$. Consider a sequence $\{t_n\}_{n=1}^{\infty}$ defined as follows: set $t_1 = 1$ and if t_{n-1} is defined, then choose t_n be the unique integer s.t.

$$\frac{t_{n-1}}{c} - 1 < t_n \leq \frac{t_{n-1}}{c}.$$

By Dirichlet's approximation theorem we find a pair of positive integers (x_n, y_n) for each t_n s.t. $1 \leq y_n \leq t_n$ and

$$\left|\frac{x_n}{y_n} - \sqrt{D}\right| < \frac{1}{y_n^2}.$$

Note that the choice of t_n forces the sequence y_n to be increasing. Then also

$$\left|\frac{x_n}{y_n} + \sqrt{D}\right| < \frac{1}{y_n^2} + 2\sqrt{D}$$

so

$$\left|\frac{x_n^2}{y_n^2} - D\right| = \left|\frac{x_n}{y_n} - \sqrt{D}\right| \cdot \left|\frac{x_n}{y_n} + \sqrt{D}\right| < \frac{1}{y_n^2} \cdot \left(\frac{1}{y_n^2} + 2\sqrt{D}\right).$$

Thus

$$|x_n^2 - Dy_n^2| < 1 + 2\sqrt{D}$$

for every $n \in \mathbb{N}$. In particular, every pair (x_n, y_n) is a solution for one of the equations $x_n^2 - Dy_n^2 = \ell$. $\ell \in \{-\lfloor 2\sqrt{D} + 1\rfloor, ..., \lfloor 2\sqrt{D} + 1\rfloor\} \setminus \{0\}$. By the pigeon-hole principle some $M = \lfloor 2\sqrt{D} + 1\rfloor^2 + 1$ out of solutions (x_n, y_n) for $n = 1, ..., 2\lfloor 2\sqrt{D} + 1\rfloor^3 + 1$ satisfy some equation $x^2 - Dy^2 = k$. Now, like in the page 72. of the lecture notes, we find a solution

$$\left(\frac{x'y'' - Dy'x''}{k}\right)^2 - D\left(\frac{y'x'' - y''x'}{k}\right)^2 = 1.$$

Now the minimal positive y s.t. $x^2 - Dy^2 = 1$ is

$$\begin{aligned}
&\leq \left|\frac{y'x'' - y''x'}{k}\right| \\
&\leq y'x'' \\
&\leq y_M(1 + y_M\sqrt{D}) \\
&\leq t_M(1 + t_M\sqrt{D}) \\
&\leq \frac{1}{c^M}\left(1 + \frac{\sqrt{D}}{c^M}\right) \\
&= (4\sqrt{D})^M\left(1 + 4^M D^{(M+1)/2}\right) \\
&= (4\sqrt{D})^{\lfloor 2\sqrt{D}+1\rfloor^2+1}\left(1 + 4^{\lfloor 2\sqrt{D}+1\rfloor^2+1} D^{(\lfloor 2\sqrt{D}+1\rfloor^2+1)/2}\right) \\
&=: \phi(D).
\end{aligned}$$

This concludes the proof. □