INTRODUCTION TO NUMBER THEORY. (Fall 2015)

9. EXERCISES (Mo 23.11)

1. Can the following integers be expressed as sums of two squares? If so, in how many ways?

(i) $n = 90$ (ii) $n = 2331$.

2. (i) Why are the two factorisations $10 = (3+i)(3+i)$ and $10 = (1+3i)(1-3i)$ not a contradiction to the unique (Gaussian) prime factorisation theorem?

(ii) The same question for $13 = (2+3i)(2-3i) = (3-2i)(3+2i)$.

3. In the lectures the number $r_2(n)$ was defined as the number of integer pairs (x, y) such that

$$x^2 + y^2 = n.$$

Show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} r_2(k) = \pi.$$

by observing that the sum on the right hand side is the number of integer points (i.e. elements in $\mathbf{Z}^2$) inside a closed disc centered at origin and of radius $\sqrt{n}$.

4. Let n be a positive integer, whose (standard) prime factorisation is

$$n = 2^e p_1^{\alpha_1} \cdots p_k^{\alpha_k} q_1^{\beta_1} \cdots q_\ell^{\beta_\ell},$$

where the p_i:s and q_i:s are unequal odd primes with $p_i \equiv 1 \pmod 4$ $(1 \le i \le k)$ ja $q_s \equiv -1 \pmod 4$ $(1 \le s \le \ell)$. How many Gaussian factors does n have?

5. Try to develop any kind of 'method', by using our knowledge on Gaussian primes to find all Gaussian prime factors of a given Gaussian integer λ. Apply it to the example $\lambda = 7 + i$ considered in the lectures.

6. Try to apply Gaussian integers to reprove our formula for Pythagorean triples.

7*. Try to say something on the following question: assume $\alpha \in Z[i]$ is given. How many different residue classes (mod α) there exists?

8.** (i) Solve the equation $x^2 + 4 = y^3$ in integers.

(ii) Solve the equation $x^5 - 1 = y^2$ in integers.

Hints:

E.1: [Trial or alternatively the theorems of the lectures.]

T.6: [E.g. start by (again) reducing to the case where z is odd and by writing down the Gaussian prime decompositions of $(x + iy)$ and z and apply the uniqueness of prime decompositions. Or, look at net/wikipedia to find other kind of shortcuts.]

T.7: [E.g. try to use geometric interpretation]

T.8: [Gaussian integers might be helpful!]