INTRODUCTION TO NUMBER THEORY. (Fall 2015)

7. EXERCISES (Mo 09.11)

1. Find the continued fraction development of the positive solution of the second degree equation $x^2 = 1 + 7x$. How many partial denominators are needed so that one obtains the accuracy of 10 desimals?

2. Use induction and the recursion formulas for partial numerators p_n and partial denominators q_n to prove part (ii) of Theorem 5.8, i.e.

$$q_{n-1}p_n - q_n p_{n-1} = (-1)^{n-1} \qquad \text{for all } n \geq 1.$$

3. (i) Show that if $c > 0$ is a fixed constant and α is rational, then the inequality $|\alpha - \frac{p}{q}| \leq cq^{-2}$ can hold only for finitely many *distinct* rationals p/q.

(ii) Use part (i) and the estimate $|\alpha - p_n/q_n| \leq q_n^{-2}$ for convergence p_n/q_n of the continued fraction expansion of α to deduce that the value of a simple infinite continued fraction is always irrational.

4. Develop the *golden ratio* $\alpha = (1 + \sqrt{5})/2$ in continued fractions. Show that the n :th convergent is F_{n+2}/F_{n+1}, where the *Fibonacci numbers* F_n are defined using the recursion $F_0 = 0$, $F_1 = 1$ ja $F_{k+2} = F_{k+1} + F_k$ kun $k \geq 0$. Try to give (perhaps heuristic) grounds for the claim that the golden ratio is the number that is most difficult to approximate by rational numbers!

5. Find the continued fractions of the following numbers

(i) $\sqrt{11}$, (ii) $\sqrt{13}$.

6. Using the answer to the previous exercise, find the fundamental solutions of Pell's equations

(i) $x^2 - 11y^2 = 1$, (ii) $x^2 - 13y^2 = 1$.

7. Show that the partial denominators λ_n of the continued fraction expansion of a given irrational number x are uniformly bounded if and only if there is a constant $c > 0$ so that for all rational numbers p/q it holds that

$$|x - \frac{p}{q}| \geq \frac{c}{q^2}.$$

8*. Let $\frac{p_n}{q_n}$ be the n:th convergent of the infinite (simple) continued fraction $\alpha = \{\lambda_0; \lambda_1, \lambda_2, \ldots\}$.

(i) Prove the inequality $q_n \leq 2^n \prod_{k=1}^{n} \lambda_k$

(ii) Show that β is transcendental, where $\beta := \{0; 2^{1!}, 2^{2!}, 2^{3!}, \ldots\}$.

Hints:

E.1: [Follow the examples in the lectures. For the accuracy statement, you may use computer or try to use Cor. 5.12 of the lectures.]

E.3: [Recall the basic inequality for two *different* rational numbers we have $|\frac{p}{q} - \frac{p'}{q'}| \geq \frac{1}{qq'}$.]

E.7: [Recall Cor. 5.12 of the lectures.]