INTRODUCTION TO NUMBER THEORY. (Fall 2015)

6. EXERCISES (Mo 02.11)

1. Find the smallest (positive) solution to Pell's equation $x^2-Dy^2=1$ for $D\in\{2,3,5,6,7,8,10\}$.

2. Show directly (without using Liouville's theorem) that $\left|\frac{p}{q}-\sqrt{3}\right|\geq\frac{1}{10q^2}$ for all rational numbers p/q.

3. Verify that the number $\sqrt{2}-\sqrt[3]{3}$ is an algebraic number by finding a polynomial equation with integer coefficients that it satisfies.

4. Find the continued fraction representations of the numbers

(i) $\frac{57}{111}$ (ii) $\sqrt{3}$ (ii) $(\sqrt{5}+1)/2$ (ii) e

[Hint: do the last one by computer]

5. Show that second order algebraic (real) numbers are exactly the numbers of the form

$$x=(a+\sqrt{D})/b$$

where a,b,D are integers such that $b\neq 0$ and $D\geq 2$ is not a square.

6. (i) Let (x_1,y_1), $(x_2,y_2),\ldots$ be the positive solutions of Pell's equation $x^2-Dy^2=1$ in increasing order. Show that they satisfy the recursion

$$\begin{cases} x_{k+1}=ax_k+by_k \\ y_{k+1}=cx_k+dy_k, \end{cases}$$

where a,b,c,d are suitable integers .

(ii) Show that the sequence (x_k) satisfies the recursion

$$x_{k+1}=2x_1x_k-x_{k-1}.$$

What is the corresponding formula for (y_k) ?

7. Show that the Liouville numbers

$$\xi=1\pm\frac{1}{2^{1!}}\pm\frac{1}{2^{2!}}\pm\frac{1}{2^{3!}}\pm\frac{1}{2^{4!}}\pm\ldots$$

are not rational.

8*. Try to make the proof given in the lectures quantitative, i.e. find some concrete function ϕ such that $y_1\leq\phi(D)$, where y_1 is the smallest positive solution of Pell's equation $x^2-Dy^2=1$.

Hints:

E.2: [Use the identity $(\sqrt{3} - p/q)(\sqrt{3} + p/q) = 3 - p^2/q^2$ and multiply by q^2.]

E.6: [Use the representation given by Thm 5.4]

E.7: [Observe that for two *different* rational numbers we have $|\frac{p}{q} - \frac{p'}{q'}| \geq \frac{1}{qq'}$.]