1 Introduction

This notes are meant to provide a conceptual background for the numerical construction of random variables. They can be regarded as a mathematical complement to the article [1] (please follow the hyperlink given in the bibliography). You are strongly recommended to read [1]. The recommended source for numerical implementation of random variables is [3] (or its FORTRAN equivalent), [2] provides and discusses analogous routines in PASCAL.

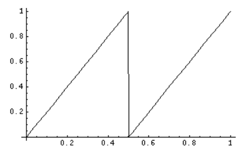
2 Binary shift map

The binary shift is the map

$$\sigma : [0, 1] \to [0, 1]$$

such that $x_{n+1} = \sigma(x_n)$ is

$$x_{n+1} = 2\, x_n \mod 1 \equiv \begin{cases} 2\, x_n & 0 \le x_n < \frac{1}{2} \\ 2\, x_n - 1 & \frac{1}{2} \le x_n \le 1 \end{cases} \tag{2.1}$$



Adopting the binary representation for any $x \in [0, 1]$

$$x = \sum_{i=1}^{\infty} a_i 2^{-i} \qquad \Rightarrow \qquad \boldsymbol{a} = (a_1, a_2, \dots)$$

where

$$a_i \in \{0, 1\} \qquad \forall\, i \in \mathbb{N}$$

we see that

- if $0 \le x < 1/2$ then $a_1 = 0$ and

$$2 \sum_{i=2}^{\infty} a_i 2^{-i} = \sum_{i=1}^{\infty} a_{i+1} 2^{-i} \qquad \Rightarrow \qquad \sigma \circ (0, a_2, a_3 \dots) = (a_2, a_3, \dots)$$

- if $1/2 \le x < 1$ then $a_1 = 1$ and

$$2 \sum_{i=1}^{\infty} a_i 2^{-i} - 1 = a_1 - 1 + \sum_{i=1}^{\infty} a_{i+1} 2^{-i} = \sum_{i=1}^{\infty} a_{i+1} 2^{-i} \qquad \Rightarrow \qquad \sigma \circ (a_1, a_2, a_3 \dots) = (a_2, a_3, \dots)$$

having used the notation

$$\sigma \circ \boldsymbol{a} \equiv \sigma(\boldsymbol{a})$$

Thus, (2.1) acts on any initial condition $x \in [0, 1]$ $x \sim (a_1, a_2, \dots)$ by removing the first entry and shifting to the left the ensuing ones. There are relevant consequences:

- The sensitive dependence of the iterates of σ on the initial conditions. If two points x and x' differ only after their n-th digit a_n , i.e. $x = (a_1, \dots, a_n, a_{n+1}, \dots)$ and $x' = (a_1, \dots, a_n, a'_{n+1}, \dots)$ this difference becomes amplified under the action of σ:

$$\sigma^n(x) = (a_{n+1}, \dots) \qquad \& \qquad \sigma^n(x') = (a'_{n+1}, \dots)$$

  where $\sigma^2(x) = \sigma(\sigma(x))$, etc.

- The sequence of iterates $\sigma^n(x)$ has the same random properties as successive tosses of a coin. Namely, $\sigma^n(x)$ is smaller or larger than $1/2$ depending on whether a_{n+1} is zero or one. If we associate to coin tossing a Bernoulli variable

$$\xi : \Omega \to \{0\,,1\} \qquad \xi(H) = 1$$

we can always associate to any realization of the sequence of i.i.d. $\{\xi_i\}_{i=1}^n$ ($\xi_i \stackrel{d}{=} \xi$) an binary sequence specifying an $x \in [0,1]$. In other words we have for any $x \in [0,1]$ an isomorphism of the type

$$x \sim \left\{ \begin{array}{lllllll} (0 & 1 & 0 & 1 & 1 & 0 & \dots) \\ (C & H & C & H & H & C & \dots) \end{array} \right.$$

- All dyadic rational numbers i.e. rational numbers of the form

$$\frac{p}{2^a} \qquad p\,, a\,, \in \mathbb{N}$$

have a terminating binary numeral. This means that the binary representation has a finite number of terms after the radix point e.g.:

$$\frac{3}{2^5} = 0.00011 \tag{2.2}$$

This means that the set of dyadic rational numbers is the basin of attraction of the fixed point in zero.

- Other rational numbers have binary representation, but instead of terminating, they recur, with a finite sequence of digits repeating indefinitely (i.e. they comprise a *periodic part* which may be preceded by a pre-periodic part):

$$\frac{13}{36} = 0.01\overline{011100}_2$$

where $\bar{\bullet}$ denotes the periodic part: they correspond to the set of *periodic orbits* together with their basin of attraction of the shift map.

- Binary numerals which neither terminate nor recur represent irrational numbers. Since rational are *dense* on real for any $x \in [0,1]$ and any ε there is at least one point on a periodic orbit (and actually an infinite number of such points) in $[x-\varepsilon, x+\varepsilon]$. This fact has important consequences for numerics. Rational numbers form (and therefore initial conditions for periodic orbits of the binary shift) a *countable infinite set* with zero Lebesgue measure. Generic initial conditions (i.e. real number on $[0,1]$) are *uncountable* and have full Lebesgue measure. Non-periodic orbits are hence in principle *generic*. Not in practice, though, if by that we mean a numerical implementation of the shist map. Computer can work only with finite accuracy numbers: at most they can work with recurring sequences of large period.

Definition 2.1 (*Perron-Frobenius operator*)**.** *Given a one dimensional map*

$$f : [0,1] \to [0,1]$$

and a probability density ρ over $[0,1]$, the one step-evolution ρ' of ρ with respect to f is governed by the Perron-Frobenius operator defined by

$$\rho'(x) = \mathcal{F}[\rho](x) := \int_0^1 dy\, \delta(x - f \circ y)\, \rho(y)$$

The definition of the Perron-Frobenius operator, allows us to associate to any map

$$x_{n+1} = f(x_n)$$

an evolution law for densities

$$\rho_{n+1}(x) = \int_0^1 dy\, \delta(x - f \circ y)\, \rho_n(y)$$

In particular we have

Definition 2.2 (*Stationary density*). *A density is stationary with respect to f if*

$$\rho(x) = \int_0^1 dy\, \delta(x - \sigma \circ y) \rho(y)$$

For the shift map we have

Proposition 2.1 (*Invariant density*). *The uniform distribution $\rho(x) = 1$ is the unque invariant density of the shift map on the space of smooth densities*

Proof. Using the definition of stationary density and the expression of shift map we have

$$\rho(x) = \int_0^{\frac{1}{2}} dy\, \delta(x - 2\, y)\, \rho(y) + \int_{\frac{1}{2}}^1 dy\, \delta(x - 2\, y + 1)\, \rho(y)$$

For any $x \in [0, 1]$ the integral gives

$$\rho(x) = \frac{1}{2}\rho\left(\frac{x}{2}\right) + \frac{1}{2}\rho\left(\frac{x+1}{2}\right) = \frac{1}{2}\sum_{i=0}^{1} \rho\left(\frac{x+i}{2}\right)$$

The equality is readily satisfied by setting $\rho(x) = 1$. The solution is also unique. We can establish using the the expression of a generic intial density after n-iterations. In order to determine such an expression we can proceed by induction

- After two steps

$$\rho^{(2)}(x) = \frac{1}{2}\left(\frac{1}{2}\rho\left(\frac{x}{4}\right) + \frac{1}{2}\rho\left(\frac{x+1}{4}\right)\right) + \frac{1}{2}\left(\frac{1}{2}\rho\left(\frac{\frac{x}{2}+1}{2}\right) + \frac{1}{2}\rho\left(\frac{\frac{x+1}{2}+1}{2}\right)\right) = \frac{1}{4}\sum_{i=0}^{3} \rho\left(\frac{x+i}{4}\right) \quad (2.3)$$

- We may infer that after n steps

$$\rho^{(n)}(x) = \frac{1}{2^n}\sum_{i=0}^{2^n-1} \rho\left(\frac{x+i}{2^n}\right) \quad (2.4)$$

- the inference implies that

$$\rho^{(n+1)}(x) = \frac{1}{2^{n+1}}\sum_{i=0}^{2^n-1} \rho\left(\frac{x+i}{2^{n+1}}\right) + \frac{1}{2^{n+1}}\sum_{i=0}^{2^n-1} \rho\left(\frac{\frac{x+i}{2^n}+1}{2}\right) \quad (2.5)$$

  The first sum ranges from $x/2^{n+1}$ to $(x+2^n-1)/2^{n+1}$. The second from $(x+2^n)/2^{n+1}$ to $(x+2^{n+1}-1)/2^{n+1}$. Therefore we can re-write the (2.5) as

$$\rho^{(n+1)}(x) = \frac{1}{2^{n+1}}\sum_{i=0}^{2^{n+1}-1} \rho\left(\frac{x+i}{2^{n+1}}\right) \quad (2.6)$$

  which proves the that the inference is correct for any n.

In the limit $n \uparrow \infty$ the latter converges to

$$\lim_{n\uparrow\infty} \rho_n(x) = \lim_{n\uparrow\infty} \frac{1}{2^n} \sum_{i=0}^{2^n-1} \rho\left(\frac{x+i}{2^n}\right) = \int_0^1 dy\, \rho(y) = 1$$

$\square$

References

[1] J. Ford. How random is a coin toss? *Physics Today*, 36:4047, 1983.

[2] P. E. Kloeden, E. Platen, and H. Schurz. *Numerical solution of SDE through computer experiments*. Universitext. Springer, 1994.

[3] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical recipes in C: the art of scientific computing*. Cambridge University Press, 1999.