INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 9

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1. Interacting Bar Magnets

Consider the following equations for angles θ_1, θ_2:

$$\dot{\theta}_1 = K\sin(\theta_1 - \theta_2) - \sin\theta_1$$
$$\dot{\theta}_2 = K\sin(\theta_2 - \theta_1) - \sin\theta_2$$

The system can be though of as a description of two thin bar magnets with a common fixed pin point in their center, such that their north (or south) pole can only rotate on a circle. In addition the magnets are subject to a magnetic field. The relevant forces are the repulsion between the poles of the two magnets, represented by the term $K\sin(\theta_i - \theta_j)$, and the force due to the magnetic field, represented by the term $\sin\theta_i$ $(i = 1, 2)$.

(1) Find and classify all fixed points of the system.
(2) Find and classify the bifurcations for varying K.
Hint: $\sin(a - b) = \cos b \sin a - \sin b \cos a$.
(3) Show that the system has no periodic orbit. You may show first that there is a function V of θ_1 and θ_2 such that $\dot{\theta}_i = -\partial V/\partial\theta_i$ (i. e., it is a gradient system).
(4) Sketch the phase portrait for $0 < K < \frac{1}{2}$ and for $K > \frac{1}{2}$.

2. Odell's Predator-Prey Model

Consider the following system

$$\dot{x} = x\big(x(1 - x) - y\big), \qquad \dot{y} = y(x - a)$$

with a constant $a \geq 0$. This is meant to describe the evolution of the population of prey $x \geq 0$ and of predators $y \geq 0$. In particular, you may restrict your attention to positive x and y.

(1) Sketch the nullclines. Find the fixpoints and classify them.
(2) Sketch the vector field for $a > 1$. Show that in this case the predators go extinct.
(3) Show that at $a_c = \frac{1}{2}$ a Hopf bifurcation occurs. Is it sub- or supercritical?
(4) Estimate the frequence of the limit cycle oscillations for a near a_c.
(5) Sketch all vector fields for different values of $0 < a < 1$ that are topologically different.

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3. A Degenerated Hopf Bifurcation

Consider the damped Duffin oscillator $\ddot{x} + \mu\dot{x} + x - x^3 = 0$.

(1) Show that the origin changes from a stable to an unstable spiral as μ decreases and changes sign.
(2) Plot the phase portrait for $\mu > 0$, $\mu = 0$, and $\mu < 0$. Show that the bifurcation is a degenerated version of a Hopf bifurcation.