INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 7

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1. A Liénard System

For a constant $\mu \in \mathbb{R}$ consider the system $\ddot{x} + \mu(x^2 - 1)\dot{x} + \tanh x = 0$.

(1) Use a computer to explore the behaviour for different values of μ.
(2) Prove that the system has a unique stable limit cycle if $\mu > 0$.
(3) What about $\mu < 0$? What about $\mu = 0$?

2. Simple Regular Perturbation

Consider the system $\ddot{x} + x = \varepsilon$ for some small constant ε.

(1) Solve the problem exactly.
(2) Solve the problem in regular (one time) perturbation theory, i. e., find x_0, x_1, and x_2 such that the solution has the form

$$x(t, \varepsilon) = x_0(t) + \varepsilon\, x_1(t) + \varepsilon^2\, x_2(t) + \mathcal{O}(\varepsilon^3).$$

What about secular terms?

3. Kosinus Pendulum

Consider the oscilator (pendulum) given by $\ddot{x} + \sin x = 0$ with a small initial amplitude $x(0) = a$, initially at rest $\dot{x}(0) = 0$. In this exercise we will compute the frequence of the cycle in two different ways. The first approach is via a conserved quantity:

(1) Consider the corresponding two-dimensional system in x and $v := \dot{x}$. Find a conserved quantity and sketch the vector field. What type of fixed point is origin?
(2) Determine the approximate shape for a cycle for a small initial amplitutes a.
(3) What is the frequence of the cycle?

To practise perturbation theory we also follows a different approach by approximating the solution in a perturbation theory with two time scales (averaged equation).

(4) What do you choose as parameter for the perturbation?
(5) Derive perturbative equations with two time scales.
(6) Determine the frequency of the cycle approximately.

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4. Method of Averaging

Consider the equation $\dot{x} = -\varepsilon\, x\, \sin^2 t$ with a constant $0 \leq \varepsilon \ll 1$.

(1) Solve the equation exactly.

(2) Now we consider the average

$$\bar{x}(t) := \frac{1}{2\pi} \int_{t-\pi}^{t+\pi} x(\tau)\, d\tau.$$

Show that $x(t) = \bar{x}(t) + \mathcal{O}(\varepsilon)$.

(3) Find an approximate differential equation satisfyed by $\bar{x}$.

(4) Solve it. Compare the obtained solution for the approximation $\bar{x}$ with the exact solution obtained for x.