INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 6

P. MURATORE-GINANNESCHI, K. SCHWIEGER

1. Epidemic

The following simple model for the evolution of an epidemic was proposed by Kermack and McKendrick in 1972. It consider the number of healthy people x and the number of sick people y subject to the equation

$$\dot{x} = -kxy, \qquad \dot{y} = kxy - ly$$

with constants $k, l > 0$.

(1) Discuss the equation in the group. Try to argue for the terms but also keep in mind that certain effects are not represented in the equation.
(2) Find and classify all fixed points.
(3) Sketch the nullclines and the vector field.
(4) Find a conserved quantity. It may help to find a differential equation of dy/dx and solve it.
(5) Under what initial condition (x_0, y_0) does y increase initially. (This could be called an *epidemic*.) What happens as $t \to \infty$?

2. Vector Fields of Gradient Systems

Show that the trajectory of a gradient system $\dot{\mathbf{x}} = -\nabla V$ always crosses any equipotential $\{\mathbf{x} \mid V(\mathbf{x}) = \alpha\}$ with $\alpha \in \mathbb{R}$ at a right angle.
Sketch the vector field for the following gradient systems with

(1) $V(x, y) = x^2 + y^2$, (2) $V(x, y) = x^2 - y^2$, (3) $V(x, y) = e^x \sin y$.

3. Is it a Gradient System?

Find the gradient for the following system

$$\dot{x} = y^2 + y\cos x, \qquad \dot{y} = 2xy + \sin x,$$

that is, find a function $V : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\dot{\mathbf{x}} = -\nabla V$.
Optional: Why is this system indeed a gradient system?

4. Limit Cycle

Consider the following system

$$\dot{x} = x - y - x(x^2 + 5y^2), \qquad \dot{y} = x + y - y(x^2 + y^2).$$

(1) The origin is clearly a fixed point. What kind of fixed point?
(2) Rewrite the system in polar coordinates.
Hint: Recall that for the radius r and the angle θ you have $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

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(3) Determine the maximal radius r_0 and the minimal radius R_0 such that no trajectories can leave the corresponding annulus $\{(x,y) \mid r_0^2 \leq x^2 + y^2 < R_0^2\}$.
(4) Conclude that the system has a limit cycle within the annulus.

5. Approximate Limit Cycle

For a constant $\mu > 0$ consider the equation $\ddot{x} + \mu\, f(x)\, \dot{x} + x = 0$ with the **non-smooth** function f given by $f(x) = -1$ whenever $|x| < 1$ and $f(x) = 1$ otherwise.

(1) Show that the system can be equivalently transformed into
$$\dot{x} = \mu(y - F(x)), \qquad \dot{y} = -x/\mu$$
with the continuous, piecewise linear function
$$F(x) := \begin{cases} x+2 & , x \leq -1, \\ -x & , |x| \leq 1, \\ x-2 & , x \geq 1. \end{cases}$$
(2) Sketch the nullclines and the vector field.
(3) Show that the system exhibits relaxation oscillations for $\mu \gg 1$.
(4) For $\mu \gg 1$ find the limit cycle and estimate its period.