INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 11

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1. Poincaré Section

Consider the dynamical system in polar coordinates given by

$$\dot{r} = r - r^2, \qquad \dot{\theta} = 1.$$

(1) Compute the Poincaré map for the section given by the positive x-axis.
(2) Show that the system has a unique periodic orbit. What is its stability?

2. Discrete Dynamics

For the following discrete time dynamical systems, determine the limit behaviour for various initial conditions?

(1) $x_{n+1} := \sqrt{x_n}$, (2) $x_{n+1} := x_n^3$, (3) $x_{n+1} := \tan x_n$,
(4) $x_{n+1} := \sinh x_n$, (5) $x_{n+1} := \tanh x_n$.

3. Cubic Map

Consider the map $x_{n+1} := 3x_n - x_n^3$.

(1) Find all fixed points and classify their stability.
(2) What happens in the long time limit for $x_0 := 1.9$? What happens for $x_0 := 2.1$? Prove your observation.

4. Newton's Method

Newton's method is an algorithm to numerically compute solutions of an equation $g(x) = 0$ for a given function $g : \mathbb{R} \to \mathbb{R}$. The basic idea is to take an approximation x_n of the solution, replace g by its linearization around x, that is by $g(x) \approx g(x_n) + g'(x_n) \cdot (x - x_n)$, and solve the resulting approximate equation to obtain a better approximation x_{n+1}.

(1) Write down the map for $x \mapsto x_{n+1}$. In the following we call it *Newton map.*
(2) Now consider the particular case $g(x) = x^2 - 4$ (that is, we want to determine the roots of 4). Compute a few iterates starting from $x_0 := 1$. Notice the fast convergence.
(3) Show that the Newton has indeed two fixed points and that both are superstable.
(4) Now consider a general function g. Find simple sufficient conditions on g such that the fixed points of the Newton map are superstable.

Indeed in many practical cases Newton's method converges very quickly **if it converges at all.**

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