INTRODUCTION TO DYNAMICAL SYSTEMS AND CHAOTIC SYSTEMS

EXERCISE 10

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1. Attractors?

Consider the following (familiar) system in polar coordinates

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1$$

(1) Is the disk $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ an invariant set?
(2) Does D attract an open neighbourhood.
(3) Is D an attractor? If not, why not? If D is an attractor, then find its basin of attraction.
(4) Is the circle $C := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ an attractor? Why not / What is its basin of attraction?

2. Trapping the Lorenz System in a Ball

Consider the Lorenz system with the usual parameters $b, \sigma, r > 0$. Show that there is a ball or the form

$$x^2 + y^2 + (z - r - \sigma)^2 \leq C$$

such that all trajectories eventually enter the ball and then remain inside.

3. Numerical Experiments

Take $\sigma = 10$ and $b = 8/3$. For each of the following fixed values of r, use the computer to explore the dynamics of the Lorenz system. In particular explore the consequences of choosing different initial conditions. You may also want to look at the trajectory only after an initial calibration time.

(1) $r = 10$, (2) $r = 22$ (3) $r = 24.5$
(4) $r = 100$, (5) $r = 126.52$, (6) $r = 166.3$,
(7) $r = 212$ (8) $r = 400$.

4. Lorenz Equations for Large r

(1) Put $\varepsilon := 1/\sqrt{r}$. Find a change of variables such that for $r \to \infty$ (equivalently $\varepsilon \to 0$) the Lorenz equations become

$$\dot{X} = Y, \qquad \dot{Y} = -XZ, \qquad \dot{Z} = XY.$$

(2) Find a conserved quantity for the system above.
(3) Show that the system is volume preserving.

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