Introduction to Continuous Logic
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Exercise 3

1. Show that for a nonprincipal ultrafilter, the D-limit of a convergent sequence coincides with its ordinary limit. Show also, that nonprincipality is a necessary condition for this.

2. Let X be a topological space. Show that the following are equivalent.

(1) X is compact.
(2) For any family $(x_i)_{i\in I}$ in X and any ultrafilter D on I, the D-limit $\lim_{i,D} x_i$ exists.

3. Let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be bounded families of reals and let D be an ultrafilter on I. Show that

(1) for any $r, s \in \mathbb{R}$, $\lim_{i,D}(ra_i + sb_i) = r \lim_{i,D} a_i + s \lim b_i$,
(2) $\lim_{i,D}(a_i \cdot b_i) = \lim_{i,D} a_i \cdot \lim_{i,D} b_i$.

4. Let (M_i, d_i), $i \in I$, be bounded metric spaces, all of diameter $\leq K$. Let D be an ultrafilter on I and define d on $\prod_{i\in I} M_i$ by

$$d((x_i)_{i\in I}, (y_i)_{i\in I}) = \lim_{i,D} d_i(x_i, y_i).$$

Show that d is a pseudometric.

5. Let (M, d) be a compact metric space. Show that the diagonal embedding of M into the ultrapower $(M)_D$ is surjective, and thus the ultrapower is isomorphic to the original M.