Introduction to Continuous Logic
Department of Mathematics and Statistics, University of Helsinki
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Exercise 2

1. Continuous logic cannot express negation, i.e., for an L-sentence φ, e.g. the expression $\varphi \leq r$ is (shorthand for) a condition, but the expression $\varphi > r$ is in general not. Still, continuous logic extends first order logic (which has negations). Explain how this is possible.

2. Assume that $\mathcal{F}$ is a full system of connectives. Show that for any $\varepsilon > 0$ and any L-formulas $\varphi(x_1, \ldots, x_n)$ there is an $\mathcal{F}$-restricted (i.e., one using as connectives only those in $\mathcal{F}$) L-formula $\psi(x_1, \ldots, x_n)$ such that for all L-structures $\mathcal{M}$ one has

$$|\varphi^{\mathcal{M}}(a_1, \ldots, a_n) - \psi^{\mathcal{M}}(a_1, \ldots, a_n)| \leq \varepsilon$$

for all $a_1, \ldots, a_n \in M$.

3. Give an axiomatization of (the unit ball of) an infinite dimensional Hilbert space.

4. A condition is *universal* if it is of the form $\sup_{\bar{x}} \varphi = 0$, where φ is quantifier-free ($\sup_{\bar{x}}$ is shorthand for $\sup_{x_1} \ldots \sup_{x_n}$). Let $\mathcal{N}$ be a submodel of $\mathcal{M}$ and assume $\mathcal{M} \models E$ for some universal E. Show that $\mathcal{N} \models E$.

5. Let $(\mathcal{M}_i)_{i<\kappa}$ be an elementary chain (i.e., $\mathcal{M}_i \preccurlyeq \mathcal{M}_j$ for $i < j$). Show that $\mathcal{M}_j \preccurlyeq \bigcup_{i<\kappa} \mathcal{M}_i$ for each $j < \kappa$, where $\bigcup_{i<\kappa} \mathcal{M}_i$ denotes the model defined on the *completion* of the union of the universes M_i.