K, L, M chain complexes

$0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \longrightarrow 0$ short exact sequence

Thus, in the following diagram all horizontal sequences are exact:

$$
\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & K_{n+1} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & L_{n+1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_{n+1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & K_n \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & L_n \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_n \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & K_{n-1} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & L_{n-1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_{n-1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & K_{n-2} \arrow[r, "\alpha"] \arrow[d] & L_{n-2} \arrow[r, "\beta"] \arrow[d] & M_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \; & \; & \; &
\end{tikzcd}
\qquad (**)
$$

Thus α and β are chain maps,

$\alpha(Z_n(K)) \subset Z_n(L)$ $\quad$ $\beta(Z_n(L)) \subset Z_n(M)$ $\quad \forall n$
$\alpha(B_n(K)) \subset B_n(L)$ $\quad$ $\beta(B_n(L)) \subset B_n(M)$

$\Rightarrow$ There are induced homomorphisms

$\alpha_* : H_n(K) \to H_n(L)$, $[z] \mapsto [\alpha(z)]$,

$\beta_* : H_n(L) \to H_n(M)$, $[z] \mapsto [\beta(z)]$, $\quad \forall n$

We show that there is a homomorphism

$\Delta_n : H_n(M) \longrightarrow H_{n-1}(K) \quad \forall\, n \in \mathbb{Z}$

induced by $(**)$ and called the <u>connecting homomorphism</u>.

We already constructed a function

$$\hat{\Delta}_n : Z_n(M) \to H_{n-1}(K), \quad m \mapsto [k]$$

where $\alpha(k) = \partial l$ and $\beta(l) = m$.

$$\begin{tikzcd} & l \in L_n \arrow[r, "\beta"] \arrow[d, "\partial"] & M_n \ni m \\ k \in K_{n-1} \arrow[r, "\alpha"] & L_{n-1} \ni \partial l & \end{tikzcd}$$

Here: β surj $\Rightarrow \exists l : \beta(l) = m$

$$\begin{tikzcd} l \in L_n \arrow[r, "\beta"] \arrow[d, "\partial"] & M_n \ni m \arrow[d, "\partial''"] \\ \partial l \in L_{n-1} \arrow[r, "\beta"] & M_{n-1} \end{tikzcd}$$

Here $\beta \partial l = \partial'' \beta l = \partial'' m = 0$, since $m \in Z_n(M)$

$\Rightarrow \left.\begin{array}{l} \partial l \in \ker \beta = \operatorname{im} \alpha \\ \alpha \text{ inj} \end{array}\right\} \Rightarrow \exists !\, k \in K_{n-1} : \alpha(k) = \partial l$ (*)

(*) <u>Notice</u>:

$$\begin{tikzcd} k \in K_{n-1} \arrow[r, "\alpha"] \arrow[d, "\partial'"] & L_{n-1} \ni \partial l \arrow[d, "\partial"] \\ K_{n-2} \arrow[r, "\alpha"] & L_{n-2} \end{tikzcd}$$

$\alpha \partial'(k) = \partial \alpha(k) = \partial \partial l = 0$

α inj $\Rightarrow \partial' k = 0 \Rightarrow k \in Z_{n-1}(K)$

Let $l_1 \in L_n$ be another element such that $\beta(l_1) = m$.
Then $\exists !\, k_1 \in K_{n-1} : \alpha(k_1) = \partial l_1$.
Then

$$\beta(l - l_1) = \beta(l) - \beta(l_1) = m - m = 0.$$

$\Rightarrow l - l_1 \in \ker \beta = \operatorname{im} \alpha \Rightarrow \exists \tilde{k} \in K_n : \alpha(\tilde{k}) = l - l_1$.

Here $\alpha \underbrace{\partial' \tilde{k}}_{\in K_{n-1}} = \partial \alpha \tilde{k} = \partial(l - l_1) = \partial l - \partial l_1 = \alpha(k) - \alpha(k_1)$
$= \alpha(k - k_1)$

α inj. $\Rightarrow \partial' \tilde{k} = k - k_1$. $\Rightarrow k - k_1 \in B_{n-1}(K)$

$\Rightarrow [k] = [k_1] \in H_{n-1}(K)$.

$\hat{\Delta}_n : Z_n(M) \to H_{n-1}(K)$ is a homomorphism:

Let $m, m' \in Z_n(M)$.

Then $\hat{\Delta}_n(m) = [k]$, where $\alpha(k) = \partial l$, $\beta(l) = m$

$\hat{\Delta}_n(m') = [k']$, where $\alpha(k') = \partial l'$, $\beta(l') = m'$.

Thus $\alpha(k+k') = \alpha(k) + \alpha(k') = \partial l + \partial l' = \partial(l+l')$

and $\beta(l+l') = \beta(l) + \beta(l') = m + m'$.

Since $\alpha(k+k') = \partial(l+l')$ and $\beta(l+l') = m+m'$,

it follows that $\hat{\Delta}_n(m+m') = [k+k']$

$= [k] + [k']$

$= \hat{\Delta}_n(m) + \hat{\Delta}_n(m')$

$\therefore$ $\hat{\Delta}_n$ is a homomorphism.

$\hat{\Delta}_n : Z_n(M) \to H_{n-1}(K)$ induces a homomorphism

$$\Delta_n : \underset{\substack{\shortparallel \\ Z_n(M)/B_n(M)}}{H_n(M)} \to H_{n-1}(K) \qquad \forall n \in \mathbb{Z},$$

if $\hat{\Delta}_n(B_n(M)) = 0 \in H_{n-1}(K)$.

Let's check that $\hat{\Delta}_n(B_n(M)) = 0$:

$$\begin{tikzcd}
 & L_{n+1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_{n+1} \arrow[d, "\partial''"] \\
 & L_n \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_n \\
K_{n-1} \arrow[r, "\alpha"] & L_{n-1} &
\end{tikzcd}$$

Let $m \in B_n(M) \subset M_n$.

$\Rightarrow \exists\, \tilde{m} \in M_{n+1} : \partial''(\tilde{m}) = m$.

β surj. $\Rightarrow \exists\, \tilde{\ell} \in L_{n+1} : \beta(\tilde{\ell}) = \tilde{m}$.

Then $\beta \partial \tilde{\ell} = \partial'' \beta \tilde{\ell} = \partial'' \tilde{m} = m$.

Also: $\partial \partial \tilde{\ell} = 0 \in L_{n-1}$. α inj $\Rightarrow$ the only element in K_{n-1} that α takes to 0 is 0.

Thus: $\alpha(0) = \partial(\partial \tilde{\ell})$ and $\beta(\partial \tilde{\ell}) = m$.

Definition of $\hat{\Delta}_n \Rightarrow \hat{\Delta}_n(m) = 0$.

Thus $\hat{\Delta}_n(B_n(M)) = 0$ and $\hat{\Delta}_n$ induces

a homomorphism

$$\Delta_n : H_n(M) \to H_{n-1}(K).$$

$$[m] \mapsto [k], \quad \text{where } \alpha(k) = \ell \text{ and } \partial \ell = m$$

Lemma 14.11

Let $0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$

be a short exact sequence of chain complexes. Then there is a long exact sequence

$$\cdots \to H_{n+1}(M) \xrightarrow{\Delta} H_n(K) \xrightarrow{\alpha_*} H_n(L) \xrightarrow{\beta_*} H_n(M) \xrightarrow{\Delta} H_{n-1}(K) \to \cdots$$

proof.

a) Exactness at $H_n(K)$:

1) $\operatorname{im} \Delta \subset \ker \alpha_*$:

Let $[m] \in H_{n+1}(M)$, $m \in Z_{n+1}(M)$.
Let $l \in L_{n+1}$ be s.t. $\beta(l) = m$ and $k \in Z_n(K)$ be s.t. $\alpha(k) = \partial l$. Then

$\Delta([m]) = \Delta_{n+1}([m]) = [k] \in H_n(K)$

and

$$(\alpha_* \circ \Delta)([m]) = \alpha_*[k] = [\alpha(k)] = [\partial l] = 0 \in H_n(L).$$

$\Rightarrow \Delta([m]) \subset \ker \alpha_*$. $\quad \therefore \operatorname{im} \Delta \subset \ker \alpha_*$

2) $\ker \alpha_* \subset \operatorname{im} \Delta$:

$$\begin{tikzcd}
 & l \arrow[d, phantom, "\in"] & \\
 & L_{n+1} \arrow[r, "\beta"] \arrow[d, "\partial"] & M_{n+1} \arrow[d, "\partial''"] \\
K_n \arrow[r, "\alpha"] & L_n \arrow[r, "\beta"] & M_n \\
k \arrow[u, phantom, "\in"] & \alpha(k) \arrow[u, phantom, "\in"] &
\end{tikzcd}$$

Let $[k] \in H_n(K)$: $\alpha_*([k]) = 0$.

$\Rightarrow [\alpha(k)] = \alpha_*([k]) = 0 \in H_n(L)$.

$\Rightarrow \alpha(k) \in B_n(L)$. $\Rightarrow \exists\, l \in L_{n+1} : \partial l = \alpha(k)$.

Write: $\beta(l) = m \in M_{n+1}$. Then

$$\partial''(m) = \partial''\beta(l) = \beta\partial(l) = \beta\alpha(k) = 0, \text{ since } \beta \circ \alpha = 0$$

$\therefore m \in Z_{n+1}(M)$

Now : $\begin{cases} \beta(l) = m \in Z_{n+1}(M) & \text{and} \\ \partial l = \alpha(k) \end{cases}$

Thus

$$\hat{\Delta}(m) = [k], \text{ i.e., } \Delta([m]) = [k].$$

$\Rightarrow [k] \in \operatorname{im}\Delta$. $\quad \therefore \quad \ker \alpha_* \subset \operatorname{im}\Delta$.

1 and 2 $\Rightarrow \operatorname{im}\Delta = \ker\alpha_*$.

b) Exactness at $H_n(L)$:

1) $\operatorname{im}\alpha_* \subset \ker\beta_*$:

$\beta\circ\alpha = 0 \Rightarrow 0 = 0_* = (\beta\circ\alpha)_* = \beta_*\circ\alpha_*$.
$\Rightarrow \operatorname{im}\alpha_* \subset \ker\beta_*$.

2) $\ker\beta_* \subset \operatorname{im}\alpha_*$:

Let $l \in Z_n(L)$, $\beta_*([l]) = 0$.

$$\begin{tikzcd}
 & \tilde{l} \in L_{n+1} \arrow[r, "\beta"] \arrow[d, "\partial"] & \tilde{m} \in M_{n+1} \arrow[d, "\partial''"] \\
k \in K_n \arrow[r, "\alpha"] \arrow[d, "\partial'"] & l \in L_n \arrow[r, "\beta"] \arrow[d, "\partial"] & M_n \ni \beta(l) \\
K_{n-1} \arrow[r, "\alpha"] & L_{n-1} &
\end{tikzcd}$$

$\Rightarrow 0 = \beta_*([l]) = [\beta(l)]$

$\Rightarrow \beta(l) \in B_n(M)$

$\Rightarrow \exists\, \tilde{m} \in M_{n+1} : \partial''(\tilde{m}) = \beta(l)$

β surj $\Rightarrow \exists\, \tilde{l} \in L_{n+1} : \beta(\tilde{l}) = \tilde{m}$

Now, $\beta\partial\tilde{l} = \partial''\beta\tilde{l} = \partial''\tilde{m} = \beta(l)$.

$\Rightarrow \beta(l - \partial\tilde{l}) = \beta(l) - \beta(\partial\tilde{l}) = 0 \quad \Rightarrow l - \partial\tilde{l} \in \ker\beta = \operatorname{im}\alpha$

$\Rightarrow \exists\, k \in K_n : \alpha(k) = l - \partial\tilde{l}$.

Then $\alpha\partial' k = \partial\alpha k = \partial(l - \partial\tilde{l}) = \partial l - \overbrace{\partial\partial\tilde{l}}^{0} = \partial l = 0$, since $l \in Z_n(L)$.

α inj $\Rightarrow \partial' k = 0. \Rightarrow k \in Z_n(K). \Rightarrow [k] \in H_n(K).$

$\alpha_*([k]) = [\alpha(k)] = [\ell - \partial \tilde{\ell}] = [\ell]$

$\therefore [\ell] \in \operatorname{im} \alpha_*$

$\therefore \ker \beta_* \subset \operatorname{im} \alpha_*$

1 and 2 $\Rightarrow \operatorname{im} \alpha_* = \ker \beta_*$

c) Exactness at $H_n(M)$:

1) $\operatorname{im} \beta_* \subset \ker \Delta$:

Let $[\ell] \in H_n(L)$, where $\ell \in Z_n(L)$.
$\Rightarrow \partial \ell = 0.$

$$\begin{tikzcd} & \ell \in L_n \arrow[r, "\beta"] \arrow[d, "\partial"] & M_n \ni m \\ K_{n-1} \arrow[r, "\alpha"] & L_{n-1} \ni \partial \ell & \end{tikzcd}$$

Write: $\beta(\ell) = m$
Then

$$\begin{cases} \beta(\ell) = m \\ \alpha(0) = 0 = \partial \ell \end{cases}$$ implies that $\Delta([m]) = 0 \in H_{n-1}(K)$.

Since $\beta_*([\ell]) = [\beta(\ell)] = [m]$, it follows that

$\Delta \beta_*([\ell]) = \Delta([m]) = 0. \quad \therefore \operatorname{im} \beta_* \subset \ker \Delta$

2) $\ker \Delta \subset \operatorname{im} \beta_*$:

Let $[m] \in \ker \Delta \subset H_n(M)$.
Then

$$\Delta([m]) = [k],$$

$$\begin{tikzcd} \tilde{k} \in K_n \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & \ell \in L_n \arrow[r, "\beta"] \arrow[d, "\partial"] & M_n \ni m \\ k \in K_{n-1} \arrow[r, "\alpha"'] & L_{n-1} & \end{tikzcd}$$

where $k \in Z_{n-1}(K)$ is s.t.
$\alpha(k) = \partial \ell$ where $\ell \in L_n$ and $\beta(\ell) = m$.

Now, $\Delta([m]) = [k] = 0 \Rightarrow k \in B_{n-1}(K)$
$\Rightarrow \exists\, \tilde{k} \in K_n : \partial'(\tilde{k}) = k.$

Then $\partial\alpha(\tilde{k}) = \alpha\partial'(\tilde{k}) = \alpha(k) = \partial l$

$\Rightarrow \partial(l - \alpha(\tilde{k})) = 0$, i.e, $l - \alpha(\tilde{k}) \in Z_n(L)$,

and $\beta(l - \alpha(\tilde{k})) = \beta(l) - \underbrace{\beta\alpha(\tilde{k})}_{=0} = \beta(l) = m$

$\Rightarrow \beta_*([l - \alpha(\tilde{k})]) = [\beta(l - \alpha(\tilde{k}))] = [m]$.

$\therefore$ $[m] \in \operatorname{im} \beta_*$

$\therefore$ $\ker \Delta \subset \operatorname{im} \beta_*$

1 and 2 $\Rightarrow$ $\ker \Delta = \operatorname{im} \beta_*$. $\square$

Proposition 14.12. Let

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K \arrow[r, "\alpha"] \arrow[d, "d"] & L \arrow[r, "\beta"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "h"] & 0 \arrow[d] \\
0 \arrow[r] & K' \arrow[r, "\alpha'"'] & L' \arrow[r, "\beta'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram, where the horizontal lines are short exact sequences of chain complexes and d, g and h are chain maps. Then the diagram

$$\begin{tikzcd}
H_n(M) \arrow[r, "\Delta"] \arrow[d, "h_*"'] & H_{n-1}(K) \arrow[d, "d_*"] \\
H_n(M') \arrow[r, "\Delta'"'] & H_{n-1}(K')
\end{tikzcd}$$

commutes.

proof. Let $[m] \in H_n(M)$. Then $m \in Z_n(M)$.
Let $\ell \in L_n$ be s.t. $\beta(\ell) = m$, and
let $k \in Z_{n-1}(K)$ be s.t. $\alpha(k) = \partial \ell$.
Then

$$\Delta([m]) = [k]$$

and

$$f_*(\Delta([m])) = f_*[k] = [f(k)] \in H_{n-1}(K').$$

On the other hand,

$$h_*([m]) = [h(m)] \in H_n(M').$$

$$\begin{tikzcd}
 & \ell \in L_n \arrow[r, "\beta"] \arrow[d, "\partial"] \arrow[dd, bend right, "g"'] & M_n \ni m \arrow[dd, "h"] \\
k \in K_{n-1} \arrow[r, "\alpha"] \arrow[dd, "f"'] & L_{n-1} \ni \partial\ell \arrow[dd, bend left, "g"] & \\
 & g(\ell) \in L'_n \arrow[r, "\beta'"] \arrow[d, "\partial"] & M'_n \ni h(m) \\
f(k) \in K'_{n-1} \arrow[r, "\alpha'"] & L'_{n-1} &
\end{tikzcd}$$

Also, $\beta' g(\ell) = h\beta(\ell) = h(m)$ and $f(k) \in Z_{n-1}(K')$, and

$$\alpha'(f(k)) = g(\alpha(k)) = g(\partial \ell) = \partial g(\ell)$$

Thus

$$\begin{cases} \beta'(g(\ell)) = h(m) \\ \alpha'(f(k)) = \partial(g(\ell)) \end{cases}$$

$$\Rightarrow \Delta' h_*([m]) = \Delta'([h(m)]) = [f(k)] = f_*([k]) = f_*\Delta([m])$$

$$\therefore \ \Delta' \circ h_* = f_* \circ \Delta. \quad \square$$

Corollary 14.13. The diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(M) \arrow[r, "\Delta"] \arrow[d, "h_*"] \arrow[dr, phantom, "1"] & H_n(K) \arrow[r, "\alpha_*"] \arrow[d, "f_*"] \arrow[dr, phantom, "2"] & H_n(L) \arrow[r, "\beta_*"] \arrow[d, "g_*"] \arrow[dr, phantom, "3"] & H_n(M) \arrow[r, "\Delta"] \arrow[d, "h_*"] & H_{n-1}(K) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(M') \arrow[r, "\Delta'"] & H_n(K') \arrow[r, "\alpha'_*"] & H_n(L') \arrow[r, "\beta'_*"] & H_n(M') \arrow[r, "\Delta'"] & H_{n-1}(K') \arrow[r] & \cdots
\end{tikzcd}$$

commutes and the horizontal lines are exact.

__proof.__ Lemma 14.11 $\Rightarrow$ the horizontal lines are exact.

Proposition 14.12 $\Rightarrow$ square 1 commutes.

Square 2: $(g\circ\alpha)_* = g_*\circ\alpha_* : H_n(K) \to H_n(L')$

$g\circ\alpha = \alpha'\circ f$ $\qquad \parallel$

$(\alpha'\circ f)_* = \alpha'_*\circ f_* : H_n(K) \to H_n(L')$

Square 3: $(h\circ\beta)_* = h_*\circ\beta_* : H_n(L) \to H_n(M')$

$h\circ\beta = \beta'\circ g$ $\qquad \parallel$

$(\beta'\circ g)_* = \beta'_*\circ g_* : H_n(L) \to H_n(M')$.

FROM

$$\begin{tikzcd}
K_n \arrow[r, "\alpha"] \arrow[d, "f"] & L_n \arrow[r, "\beta"] \arrow[d, "g"] & M_n \arrow[d, "h"] \\
K'_n \arrow[r, "\alpha'"'] & L'_n \arrow[r, "\beta'"'] & M'_n
\end{tikzcd}$$

$\square$

15. Chain homotopy

__Definition 15.1.__ Let K and K' be chain complexes and let $f: K \to K'$ and $g: K\to K'$ be chain maps. A chain homotopy between f and g is a family of homomorphisms

$$D_n : K_n \to K'_{n+1}$$

satisfying $\partial'_{n+1}\circ D_n + D_{n-1}\circ\partial_n = f - g \quad \forall n$

$$
\begin{tikzcd}
\arrow[d] & & \arrow[d] \\
K_{n+1} \arrow[d, "\partial_{n+1}"'] & & K'_{n+1} \arrow[d, "\partial'_{n+1}"] \\
K_n \arrow[d, "\partial_n"'] \arrow[uurr, "D_n"] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & K'_n \arrow[d, "\partial'_n"] \\
K_{n-1} \arrow[d] \arrow[uurr, "D_{n-1}"'] & & K'_{n-1} \arrow[d] \\
{} & & {}
\end{tikzcd}
$$

$$\partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n = f - g$$

<u>Proposition 15.2.</u> If chain maps $f, g : K \to K'$ are chain homotopic, then $f_* = g_* : H_n(K) \to H_n(K')$ for every n.

<u>proof.</u> Let $[k] \in H_n(K)$, where $k \in Z_n(K)$.
Let $\{D_n : K_n \to K'_{n+1}\}$ be a chain homotopy from f to g. Then

$$f = g + \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$$

and

$$f(k) = g(k) + \partial'_{n+1}(D_n(k)) + D_{n-1}(\underbrace{\partial_n(k)}_{=0 \text{ since } k \in Z_n(K).})$$

$$\Rightarrow f_*([k]) = [f(k)] = [g(k) + \underbrace{\partial'_{n+1}(D_n(k))}_{[\;]=0}] = [g(k)] = g_*([k]).$$

$$\therefore f_* = g_*. \quad \square$$

Let K be a chain complex. Let $id : K \to K$ and $0 : K \to K$, where $id : K_n \to K_n$ is the identity for all n and $0 : K_n \to K_n$ is the zero map for all n. Then id and 0 are chain maps.

<u>Definition 15.3.</u> Let K be a chain complex. If there is a chain homotopy from the identity map id of K to the zero chain map $0 : K \to K$, we call this chain homotopy a <u>chain contraction.</u> If there is a chain contraction of K, K is said to be <u>chain contractible.</u> If $H_n(K) = 0$, for all n, we say that K is <u>acyclic.</u>

<u>Lemma 15.4.</u> A contractible chain complex is acyclic.

<u>proof.</u> Let K be a contractible chain complex. Then $id, 0 : K \to K$ are chain homotopic. By Proposition 15.2, $(id)_* = 0_* : H_n(K) \to H_n(K)$, for every n. However, $(id)_* = id : H_n(K) \to H_n(K)$ and $0_* = 0 : H_n(K) \to H_n(K)$. Thus $0 = id : H_n(K) \to H_n(K)$, for every n, which is possible only when $H_n(K) = 0$, for every n. $\square$.

A chain complex K is called <u>free,</u> if K_n is a free abelian group for every n.

<u>Proposition 15.5.</u> Let K be a free chain complex. Then K is acyclic if and only if it is contractible.

<u>proof.</u> By Lemma 15.4, a contractible chain complex is acyclic. Assume then that K is acyclic.

$$\begin{array}{ccccccc} \to K_{n+1} & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & K_{n-1} & \xrightarrow{\partial_{n-1}} & \\ id \downarrow\downarrow 0 & & id\downarrow\downarrow 0 & & id \downarrow\downarrow 0 & & \\ \to K_{n+1} & \xrightarrow[\partial_{n+1}]{} & K_n & \xrightarrow[\partial_n]{} & K_{n-1} & \xrightarrow[\partial_{n-1}]{} & \end{array}$$

The boundary map ∂_n takes K_n onto $B_{n-1}(K) \overset{K \text{ acyclic}}{=} Z_{n-1}(K)$.

Since K_n is free, also $Z_{n-1}(K)$ is free.
Thus there is a homomorphism

$$S_{n-1} : Z_{n-1}(K) \to K_n \quad , \quad \partial_n \circ S_{n-1} = id : Z_{n-1}(K) \to Z_{n-1}(K).$$

Then

$$id_{K_n} - S_{n-1}\partial_n : K_n \to K_n, \text{ maps } K_n \text{ to } Z_n(K)$$
$$(\partial_n(id_{K_n} - S_{n-1}\partial_n)(z) = \partial_n z - \partial_n z = 0)$$

Define

$$D_n : K_n \to K_{n+1} \, , \quad D_n = S_n \circ (id_{K_n} - S_{n-1}\partial_n).$$

Then

$$\partial_{n+1} D_n + D_{n-1}\partial_n = \underbrace{\partial_{n+1} S_n}_{=id}(id_{K_n} - S_{n-1}\partial_n)$$
$$+ S_{n-1}(id_{K_{n-1}} - S_{n-2}\underbrace{\partial_{n-1})\partial_n}_{=0}$$
$$= id_{K_n} - S_{n-1}\partial_n + \underbrace{S_{n-1} id_{K_{n-1}} \partial_n}_{S_{n-1}\partial_n}$$
$$= id_{K_n}$$

Thus $\{D_n\}$ is a chain contraction. □

Example Let C be the chain complex with $C_q = 0$, if $q \neq 0,1,2$, $C_0 = \mathbb{Z}_2$ and $C_2 = C_1 = \mathbb{Z}$. The boundary maps are defined as follows:

$$\partial_2(n) = 2n$$

$$\partial_1(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

Exercise (Week 8, problem 1) $\Rightarrow$ C is acyclic.

$$\begin{tikzcd}
\to 0 \arrow[r] \arrow[d, "id"] & \mathbb{Z} \arrow[r, "\partial_2"] \arrow[d, "id"] \arrow[dl, "D_2"'] & \mathbb{Z} \arrow[r, "\partial_1"] \arrow[d, "id"] \arrow[dl, "D_1"'] & \mathbb{Z}_2 \arrow[r, "\partial_0"] \arrow[d, "id"] \arrow[dl, "D_0"'] & 0 \to \arrow[d, "id"] \arrow[dl, "D_{-1}"'] \\
\to 0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_2"'] & \mathbb{Z} \arrow[r, "\partial_1"'] & \mathbb{Z}_2 \arrow[r, "\partial_0"'] & 0 \to
\end{tikzcd}$$

Assume there is a chain contraction $D : id_K \simeq 0_K$.
Then
$$\underbrace{D_{-1}\partial_0}_{0} + \partial_1 D_0 = id : \mathbb{Z}_2 \to \mathbb{Z}_2$$

Thus
$$\mathbb{Z}_2 \xrightarrow{D_0} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}_2$$

equals the identity homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$.

This is impossible, since the only homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$ is trivial. It follows that C is not contractible.

16. Relative homology groups

Let X be a topological space and let $S_*(X)$ be the singular chain complex of X.

Let $A \subset X$ and let $i : A \hookrightarrow X$ be the inclusion.

Then i induces a chain map $i_\# : S_*(A) \to S_*(X)$,

For every n, the map

$$i_\# : S_n(A) \to S_n(X), \quad \sum_T n_T T \mapsto \sum_T n_T (i \circ T),$$

is injective. $(T : \Delta_n \to A \Rightarrow i \circ T : \Delta_n \to X)$

Now, $S_*(A)$ is a subcomplex of $S_*(X)$, and

$$S_n(A) = \{ \underbrace{\sum_{r=1}^{k} n_r T_r}_{\text{finite sum}} \in S_n(X) \mid T_r(\Delta_n) \subset A \ \forall r \}$$

The corresponding quotient complex is $S_*(X)/S_*(A)$,

where $(S_*(X)/S_*(A))_n = S_n(X)/S_n(A)$

The complex $S_*(X)/S_*(A)$ is called the singular chain complex of the pair (X, A).

We obtain a short exact sequence of chain complexes:

$$0 \longrightarrow S_*(A) \xrightarrow{i} S_*(X) \xrightarrow{j} S_*(X)/S_*(A) \longrightarrow 0$$

Thus, for $\forall n$, the sequence

$$0 \longrightarrow S_n(A) \xrightarrow{i_\#} S_n(X) \xrightarrow{j} S_n(X)/S_n(A) \longrightarrow 0$$

is exact.