Proposition 12.3. The combined homomorphism

$$S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} S_{n-2}(X)$$

is a zero homomorphism, i.e, $\partial_{n-1} \circ \partial_n = 0$.

proof. Let $T : \Delta_n \to X$ be a singular n-simplex of X. Then

$$\partial_{n-1} \partial_n (T) = \partial_{n-1} \left(\sum_{j=0}^{n} (-1)^j T^{(j)} \right) = \sum_{j=0}^{n} (-1)^j \partial_{n-1} T^{(j)}$$

$$= \sum_{j=0}^{n} (-1)^j \left(\sum_{k=0}^{n-1} (-1)^k (T^{(j)})^{(k)} \right)$$

$$= \sum_{0 \le k < j \le n} (-1)^{j+k} (T^{(j)})^{(k)} + \sum_{0 \le j \le k \le n-1} (-1)^{j+k} (T^{(j)})^{(k)} \qquad (*)$$

Lemma 12.1 $\Rightarrow$ $(T^{(j)})^{(k)} = (T^{(k)})^{(j-1)}$ when $0 \le k < j \le n$. Thus the first sum in $(*)$ equals

$$\sum_{0 \le k < j \le n} (-1)^{j+k} (T^{(k)})^{(j-1)} = \sum_{0 \le k \le j-1 \le n-1} -(-1)^{(j-1)+k} (T^{(k)})^{(j-1)}$$

$=$ - the second sum in $(*)$

$(j \leftrightarrow k, \ k \leftrightarrow j-1)$ $\qquad \therefore \partial_{n-1}\partial_n(T) = 0.$

Since $\partial_{n-1}\partial_n(T) = 0$ for all $T \in Map(\Delta_n, X)$, it follows that

$$\partial_{n-1} \circ \partial_n = 0 : S_n(X) \to S_{n-2}(X). \quad \square$$

Definition 12.4. Let $c \in S_n(X)$.

1) If $\partial_n(c) = 0$, we call c a (singular) <u>n-cycle.</u>

2) If there is such $d \in S_{n+1}(X)$ that $\partial_{n+1} d = c$, we call c a (singular) <u>n-boundary.</u>

Denote: $Z_n(X) = \ker \partial_n$
$B_n(X) = \operatorname{im} \partial_{n+1}$.

Then $Z_n(X)$ and $B_n(X)$ are subgroups of $S_n(X)$.
Proposition 12.3 $\Rightarrow B_n(X) \subset Z_n(X)$ $\forall n$.

<u>Definition 12.5.</u> For each $n \geq 0$, the n^{th} (singular) <u>homology group</u> of a space X is

$$H_n(X) = Z_n(X)/B_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Let X and Y be topological spaces and let $f: X \to Y$ be continuous. Let $T: \Delta_n \to X$ be a singular n-simplex of X. Then $f \circ T: \Delta_n \to Y$ is a singular n-simplex of Y. Define a homomorphism

$$f_\#: S_n(X) \to S_n(Y)$$

by setting $f_\#(T) = f \circ T \in S_n(Y)$ for all $T \in \mathrm{Map}(\Delta_n, X)$ and extending by linearity,

$$f_\# \left(\sum_{i=1}^{k} n_i (T_i) \right) = \sum_{i=1}^{k} n_i (f \circ T), \quad \text{where } n_i \in \mathbb{Z} \ \forall i \in \{1, \dots, k\}.$$

<u>Lemma 12.6.</u> If $f: X \to Y$ is continuous, then $\partial_n f_\# = f_\# \partial_n$, i.e., for every $n \geq 0$, the diagram

$$\begin{tikzcd} S_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#"'] & S_{n-1}(X) \arrow[d, "f_\#"] \\ S_n(Y) \arrow[r, "\partial_n"'] & S_{n-1}(Y) \end{tikzcd}$$

commutes.

<u>Proof.</u> Let $T \in \mathrm{Map}(\Delta_n, X)$. Then

$$(\partial_n \circ f_\#)(T) = \partial_n(f_\#(T)) = \partial_n(f \circ T) = \sum_{j=0}^{n} (-1)^j (f \circ T \circ e^j)$$

$= f_\#\left(\sum_{j=0}^{n}(-1)^j(T\circ e^j)\right) = f_\#(\partial_n T) = (f_\#\circ\partial_n)(T)$.

Since the singular n-simplexes T generate $S_n(X)$ and since $f_\#$ and ∂_n are homomorphisms, it follows that $\partial_n\circ f_\# = f_\#\circ\partial_n$. $\square$

<u>Lemma 12.7.</u> Let $f: X\to Y$ be continuous. Then for every $n\geq 0$,

1) $f_\#(Z_n(X))\subset Z_n(Y)$,

2) $f_\#(B_n(X))\subset B_n(Y)$.

<u>Proof.</u>

1) $c\in Z_n(X) \Rightarrow \partial_n(c)=0 \Rightarrow \partial_n(f_\#(c)) = (\partial_n\circ f_\#)(c) \overset{12.6}{=} (f_\#\circ\partial_n)(c)$
$= f_\#(\partial_n(c)) = f_\#(0) = 0 \Rightarrow f_\#(c)\in Z_n(Y)$.

2) $c\in B_n(X) \Rightarrow \exists d\in S_{n+1}(X): c=\partial_{n+1}d$.
$\Rightarrow f_\#(c) = f_\#(\partial_{n+1}d) = (f_\#\circ\partial_{n+1})(d) \overset{12.6.}{=} (\partial_{n+1}\circ f_\#)(d)$
$= \partial_{n+1}(f_\#(d)) \Rightarrow f_\#(c)\in B_n(Y)$. $\square$

It follows from Lemma 12.7, that $f_\#: S_n(X)\to S_n(Y)$ induces a homomorphism

$$f_*: Z_n(X)/B_n(X) \longrightarrow Z_n(Y)/B_n(Y)$$
$$c\,B_n(X) \longmapsto f_\#(c)\,B_n(Y)$$
$$c\in Z_n(X)$$

i.e., a homomorphism

$$f_*: H_n(X)\to H_n(Y).$$

If $c\in Z_n(X)$, we write

$$[c] = c\,B_n(X)\in Z_n(X)/B_n(X) = H_n(X).$$

$\uparrow$ homology class of c

If $c, c' \in Z_n(X)$, then

$$[c] = [c'] \in H_n(X) \iff c\,B_n(X) = c'\,B_n(X)$$
$$\iff c = c' + b, \quad b \in B_n(X).$$

Theorem 12.8. For each $n \geq 0$, $H_n : \mathrm{Top} \to \mathrm{Ab}$ is a functor.

proof. For continuous $f : X \to Y$, define

$$H_n(f) = f_* : H_n(X) \to H_n(Y), \quad c\,B_n(X) \mapsto f_\#(c)\,B_n(Y).$$

If $f : X \to X$ is the identity function id_X, then $f_\# = id : S_n(X) \to S_n(X)$, for all $n \geq 0$. Thus $H_n(f) : H_n(X) \to H_n(X)$ is the identity function, for all $n \geq 0$

Let $f : X \to Y$ and $g : Y \to W$ be continuous. Then $g \circ f : X \to W$ is continuous, and

$$(g \circ f)_\# = g_\# \circ f_\# : S_n(X) \to S_n(W), \quad \text{for all } n \geq 0.$$

$$\begin{array}{ccccc} S_n(X) & \xrightarrow{f_\#} & S_n(Y) & \xrightarrow{g_\#} & S_n(W) \\ T & \mapsto & f \circ T & \mapsto & g \circ (f \circ T) \end{array}$$

same

$$\begin{array}{ccc} S_n(X) & \xrightarrow{(g \circ f)_\#} & S_n(W) \\ T & \mapsto & (g \circ f)(T) \end{array}$$

Thus $g_* \circ f_* = (g \circ f)_* : H_n(X) \to H_n(W)$, for all $n \geq 0$,

i.e. $H_n(g) \circ H_n(f) = H_n(g \circ f) : H_n(X) \to H_n(W)$, for all $n \geq 0$.

$\square$

Proposition 12.9. If topological spaces X and Y are homeomorphic, then $H_n(X)$ and $H_n(Y)$ are isomorphic for all $n \geq 0$.

Proof. Let $f : X \to Y$ be a homeomorphism. Then there is a continuous function $g : Y \to X$ with $g \circ f = id_X$ and $f \circ g = id_Y$. Let $n \geq 0$. The functions f and g induce homomorphisms

$$f_* : H_n(X) \to H_n(Y) \quad \text{and} \quad g_* : H_n(Y) \to H_n(X),$$

respectively. By Theorem 12.8,

$$g_* \circ f_* = (g \circ f)_* = (id_X)_* = id : H_n(X) \to H_n(X), \quad \text{and}$$

$$f_* \circ g_* = (f \circ g)_* = (id_Y)_* = id : H_n(Y) \to H_n(Y).$$

Then f_* is an isomorphism with the inverse $(f_*)^{-1} = g_*$.

$\square$

13. Dimension axiom and examples

Example 13.1. Let $X = \{p\}$ be a one-point space. If $n \geq 0$, there is exactly one function $T_n : \Delta_n \to \{p\}$. Then $S_n(X) = \mathbb{Z}_{T_n}$. The singular chain complex of X is

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1}(\{p\}) \arrow[r, "\partial_{n+1}"] \arrow[d, equal] & S_n(\{p\}) \arrow[r, "\partial_n"] \arrow[d, equal] & S_{n-1}(\{p\}) \arrow[r, "\partial_{n-1}"] \arrow[d, equal] & \cdots \arrow[r] & S_0(\{p\}) \arrow[r] & 0 \arrow[d, equal] \\
\arrow[r] & \mathbb{Z}_{T_{n+1}} \arrow[r, "\partial_{n+1}"'] & \mathbb{Z}_{T_n} \arrow[r, "\partial_n"'] & \mathbb{Z}_{T_{n-1}} \arrow[r, "\partial_{n-1}"'] & \cdots & & S_{-1}(\{p\})
\end{tikzcd}$$

Now, $T_n : \Delta_n \to \{p\}$

$$\begin{tikzcd}
\Delta_{n-1} \arrow[r, "e^j"] \arrow[rr, bend right, "T_n \circ e^j = T_n^{(j)}"'] & \Delta_n \arrow[r, "T_n"] & \{p\}
\end{tikzcd}$$

Then $T_n^{(j)} = T_{n-1}$, for all $j \in \{0, \ldots, n\}$.

Therefore,

$$\partial_n(T_n) = \sum_{j=0}^{n} (-1)^j T_n^{(j)} = \sum_{j=0}^{n} (-1)^j T_{n-1} = \left(\sum_{j=0}^{n} (-1)^j\right) T_{n-1}$$

$$= \begin{cases} T_{n-1}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}.$$

$$\cdots \longrightarrow S_4(\{p\}) \xrightarrow[\cong]{\partial_4} S_3(\{p\}) \xrightarrow{\partial_3 = 0} S_2(\{p\}) \xrightarrow[\cong]{\partial_2} S_1(\{p\}) \xrightarrow{\partial_1 = 0} S_0(\{p\}) \xrightarrow{0} 0$$

Thus:

n even $\Rightarrow$ $\partial_n : S_n(\{p\}) \xrightarrow{\cong} S_{n-1}(\{p\})$ is an isomorphism

n odd $\Rightarrow$ $\partial_n = 0 : S_n(\{p\}) \to S_{n-1}(\{p\})$

Thus:

1) $H_0(\{p\}) = Z_0(\{p\}) / B_0(\{p\}) = S_0(\{p\}) / \operatorname{im} \partial_1$

$$= S_0(\{p\}) / 0 \cong S_0(\{p\}) \cong \mathbb{Z}.$$

2) $n \geq 1$, n odd:

$$H_n(\{p\}) = Z_n(\{p\}) / B_n(\{p\}) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

$$= S_n(\{p\}) / S_n(\{p\}) = 0.$$

3) $n \geq 1$, n even:

$$H_n(\{p\}) = Z_n(\{p\}) / B_n(\{p\}) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

$$= 0/0 = 0.$$

Thus: $H_n(\{p\}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases}.$

In particular, we proved the following:

<u>Theorem 13.2.</u> (Dimension Axiom) If X is a one-point space, then $H_n(X) = 0$, for all $n > 0$. $\square$

<u>Proposition 13.3.</u> Let X be path connected, $X \neq \emptyset$. Then $H_0(X) \cong \mathbb{Z}$.

<u>proof.</u> Consider

$$\cdots \to S_1(X) \xrightarrow{\partial} S_0(X) \to 0.$$

Here $Z_0(X) = S_0(X)$, and
$H_0(X) = Z_0(X)/B_0(X) = S_0(X)/B_0(X)$.

Let $T: \underset{\{1\}}{\underset{\shortparallel}{\Delta_0}} \to X$ be a singular 0-simplex.

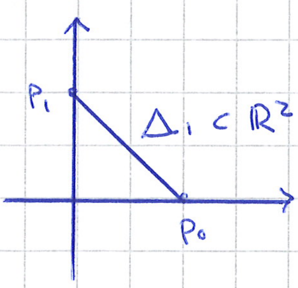
We identify T with the point $T(\Delta_0) \in X$.
Then an arbitrary element c_0 of $S_0(X)$ is of the form

$$c_0 = \sum n_x x \quad , \quad n_x \in \mathbb{Z},\ x \in X, \quad n_x \neq 0 \text{ for only finitely many } x \in X.$$

Let $T: \Delta_1 \to X$ be a singular 1-simplex.
Then

$$\begin{aligned} \partial T &= T^{(0)} - T^{(1)} \\ &= T^{(0)}(\Delta_0) - T^{(1)}(\Delta_0) \quad \text{(by identification)} \\ &= T(p_1) - T(p_0), \end{aligned}$$

where $p_1 = (0,1)$ and $p_0 = (1,0)$



Define a homomorphism

$$\eta : S_0(X) \to \mathbb{Z}, \quad \sum_{x \in X} n_x x \mapsto \sum_{x \in X} n_x \in \mathbb{Z}.$$

Since $X \neq \emptyset$, it follows that η is a surjection.

Let $T : \Delta_1 \to X$. Then

$$\eta \partial T = \eta(T(p_1) - T(p_0)) = 1 - 1 = 0.$$

Thus $B_0(X) \subset \ker \eta$.

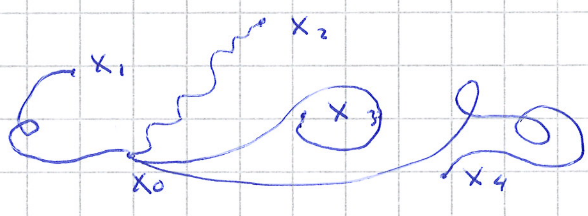
We show that $\ker \eta \subset B_0(X)$:
Let

$$c_0 = \sum_{i=1}^{k} n_i x_i \in \ker \eta,$$

then $\sum_{i=1}^{k} n_i = 0$. Let $x_0 \in X$. Let

$T_i : \Delta_1 \to X$, $T_i(p_0) = x_0$ and $T_i(p_1) = x_i$

(the T_i exist, since X is path connected).



Then

$$\partial\left(\sum_{i=1}^{k} n_i T_i\right) = \sum_{i=1}^{k} n_i \partial T_i = \sum_{i=1}^{k} n_i (T_i(p_1) - T_i(p_0))$$

$$= \sum_{i=1}^{k} n_i (x_i - x_0) = \sum_{i=1}^{k} n_i x_i - \underbrace{\left(\sum_{i=1}^{k} n_i\right)}_{0} x_0$$

$$= \sum_{i=1}^{k} n_i x_i = c_0.$$

Thus $c_0 \in B_0(X)$. It follows that $\ker \eta \subset B_0(X)$, and hence that $\ker \eta = B_0(X)$.

Since $\eta : S_0(X) \to \mathbb{Z}$ is a surjective homomorphism, there is an isomorphism,

$$\tilde{\eta} : S_0(X)/\ker\eta \cong \mathbb{Z}.$$

Therefore η induces an isomorphism

$$\bar{\eta} : H_0(X) = Z_0(X)/B_0(X) = S_0(X)/\ker\eta \cong \mathbb{Z}. \quad \square$$

<u>Definition 13.4.</u> Let $\{G_i \mid i \in I\}$ be a family of abelian groups. The direct sum $\sum_{i\in I} G_i$ of the groups G_i is the set of all $(g_i)_{i\in I}$ such that $g_i \neq 0$ for only finitely many $i \in I$.

<u>Theorem 13.5.</u> Let X be a topological space, $X \neq \emptyset$. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be the set of path components of X. Then, for every $n \geq 0$,

$$H_n(X) \cong \sum_\lambda H_n(X_\lambda).$$

<u>proof.</u> Let $\gamma = \sum n_i T_i \in S_n(X)$. For every i, the image $T_i(\Delta_n)$ is contained in a unique path component of X. Therefore, we may write $\gamma = \sum \gamma_\lambda$, where γ_λ is the sum of the terms in γ involving T_i for which $T_i(\Delta_n) \in X_\lambda$.

For every $n \geq 0$, the map $S_n(X) \to \sum_\lambda S_n(X_\lambda)$, $\gamma \mapsto (\gamma_\lambda)$, is an isomorphism. Then (check this), $\gamma \in Z_n(X)$, if and only if $\gamma_\lambda \in Z_n(X_\lambda)$ for every $\lambda \in \Lambda$, and $\gamma \in B_n(X)$, if and only if $\gamma_\lambda \in B_n(X_\lambda)$, for every $\lambda \in \Lambda$. Therefore,

$$\theta_n : H_n(X) \to \sum_\lambda H_n(X_\lambda), \quad \gamma B_n(X) \mapsto (\gamma_\lambda B_n(X_\lambda)),$$

is well defined. The inverse of θ_n is

$$\varphi_n : \sum_\lambda H_n(X_\lambda) \to H_n(X), \quad (\gamma_\lambda B_n(X_\lambda)) \mapsto (\sum \gamma_\lambda) B_n(X).$$

$\square$

14. Chain complexes

Definition 14.1. A chain complex K is a sequence

$$\cdots \longrightarrow K_{n+1} \xrightarrow{\partial_{n+1}} K_n \xrightarrow{\partial_n} K_{n-1} \xrightarrow{\partial_{n-1}} \cdots, \quad n \in \mathbb{Z}$$

where, for every $n \in \mathbb{Z}$, K_n is an abelian group, ∂_n is a group homomorphism and $\partial_n \circ \partial_{n+1} = 0$.

Often $K_n = 0$, for every $n < 0$.
Denote

$$Z_n(K) = \ker \partial_n$$
$$B_n(K) = \operatorname{im} \partial_{n+1}.$$

Since $\partial_n \circ \partial_{n+1} = 0$, it follows that $B_n(K) \subset Z_n(K)$.

Definition 14.2. The n^{th} homology group of K is

$$H_n(K) = Z_n(K) / B_n(K).$$

Definition 14.3. Let K and L be chain complexes. A chain map $f: K \to L$ is a sequence of homomorphisms $\{f_n : K_n \to L_n\}$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & K_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & K_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & L_{n+1} \arrow[r, "\partial_{n+1}"] & L_n \arrow[r, "\partial_n"] & L_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots
\end{tikzcd}$$

commutes, i.e, $\partial_n \circ f_n = f_{n-1} \circ \partial_n$, for all $n \in \mathbb{Z}$.

Let $f: K \to L$ be a chain map. Then

$$f_n(Z_n(K)) \subset Z_n(L) \quad \text{and} \quad f_n(B_n(K)) \subset B_n(L),$$

for every $n \in \mathbb{Z}$.

Thus, for every $n \in \mathbb{Z}$, we obtain a homomorphism

$$H_n(f) = f_* : H_n(K) \to H_n(L).$$
$$[z] \mapsto [f_n(z)]$$

(Here: $z \in Z_n(K)$, $[z] = z B_n(K)$, also denoted by $z + B_n(K)$, is the <u>homology class</u> of z.)

<u>Definition 14.4.</u> Chain complexes (objects) and chain maps (morphisms) form a category Comp. Composition of chain maps is defined coordinatewise: $\{g_n\} \circ \{f_n\} = \{g_n \circ f_n\}$.

Let K, L and M be chain complexes and let $f: K \to L$ and $g: L \to M$ be chain maps. Then $g \circ f : K \to M$ is a chain map and

$$\underset{\substack{\| \\ H_n(g\circ f)}}{(g\circ f)_*} = \underset{\substack{\| \\ H_n(g)}}{g_*} \circ \underset{\substack{\| \\ H_n(f)}}{f_*} : H_n(K) \to H_n(M), \text{ for all } n \in \mathbb{Z}.$$

Clearly, the identity map $id = (id_{K_n}) : K \to K$ is a chain map and it induces $H_n(id_K) = id_* = id : H_n(K) \to H_n(K)$, for every $n \in \mathbb{Z}$.

Therefore:

<u>Proposition 14.5.</u> For every $n \in \mathbb{Z}$, there is a functor H_n : Comp $\to$ Ab. $\square$

<u>Definition 14.6.</u> Let K be a chain complex. A subcomplex (more precisely, a sub chain complex) K' consists of subgroups $K'_n \subset K_n$ such that $\partial_n(K'_n) \subset K'_{n-1}$, for every $n \in \mathbb{Z}$.

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
K'_{n+1} \arrow[d, "\partial'_{n+1} = \partial_{n+1}|"'] & \subset & K_{n+1} \arrow[d, "\partial_{n+1}"] \\
K'_n \arrow[d, "\partial'_n = \partial_n|"'] & \subset & K_n \arrow[d, "\partial_n"] \\
K'_{n-1} \arrow[d] & \subset & K_{n-1} \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}$$

<u>Definition 14.7.</u> Let K be a chain complex and let K' be a subcomplex of K. The <u>quotient complex</u> K/K' is the complex

$$\cdots \longrightarrow K_{n+1}/K'_{n+1} \xrightarrow{\bar\partial_{n+1}} K_n/K'_n \xrightarrow{\bar\partial_n} K_{n-1}/K'_{n-1} \xrightarrow{\bar\partial_{n-1}} \cdots$$

where $\bar\partial_n$ is the homomorphism induced by ∂_n:

$$\bar\partial_n(c\,K'_n) = \partial_n(c)\,K'_{n-1}.$$

Exact sequences

<u>Definition 14.8.</u> A sequence

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_{n-1} \xrightarrow{\partial_{n-1}} \cdots , \qquad (*)$$

where G_n is an abelian group and ∂_n is a group homomorphism, for every $n \in \mathbb{Z}$, is called <u>exact</u>, if $\operatorname{im} \partial_{n+1} = \ker \partial_n$, for every $n \in \mathbb{Z}$.

In other words: The sequence $(*)$ is exact, if and only if, $(*)$ is a chain complex G such that $H_n(G) = 0$, for every $n \in \mathbb{Z}$.

Definition 14.9. Let A, B, C be abelian groups, and let $\alpha: A \to B$ and $\beta: B \to C$ be group homomorphisms. The sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \qquad (*)$$

is exact, if and only if,

1) α is an injection (monomorphism), and
2) β is a surjection (an epimorphism), and
3) $\operatorname{im} \alpha = \ker \beta$.

If $(*)$ is exact, it is called a <u>short exact sequence</u> of abelian groups.

Definition 14.10. Let K, L, M be chain complexes, and let $\alpha: K \to L$ and $\beta: L \to M$ be chain maps. The sequence

$$0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \longrightarrow 0 \qquad (**)$$

is called a <u>short exact sequence of chain complexes</u>, if

$$0 \longrightarrow K_n \xrightarrow{\alpha_n} L_n \xrightarrow{\beta_n} M_n \longrightarrow 0$$

is a short exact sequence of abelian groups, for every $n \in \mathbb{Z}$.

The chain map $\alpha: K \to L$ induces a homomorphism

$$\alpha_*: H_n(K) \to H_n(L), \qquad \forall\, n \in \mathbb{Z}$$

and the chain map $\beta: L \to M$ induces a homomorphism

$$\beta_*: H_n(L) \to H_n(M), \qquad \forall\, n \in \mathbb{Z}.$$

We will show that (**) induces also homomorphisms

$$\Delta_n : H_n(M) \longrightarrow H_{n-1}(K), \quad \forall\, n \in \mathbb{Z}.$$

The maps Δ_n are called <u>connecting homomorphisms.</u>

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & K_{n+1} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & L_{n+1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_{n+1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & K_n \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & L_n \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_n \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & K_{n-1} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & L_{n-1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & M_{n-1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & K_{n-2} \arrow[r, "\alpha"] \arrow[d] & L_{n-2} \arrow[r, "\beta"] \arrow[d] & M_{n-2} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

First, we construct a homomorphism

$$\hat{\Delta}_n : Z_n(M) \longrightarrow H_{n-1}(K)$$

as follows: Let $m \in Z_n(M)$, i.e., $m \in M_n$ and $\partial''(m) = 0$. β surj. $\Rightarrow \exists\, \ell \in L_n : \beta(\ell) = m$.

Then $\beta\partial(\ell) = \partial''\beta(\ell) = \partial''(m) = 0. \Rightarrow \partial(\ell) \in \ker\beta$.

Since the sequence

$$0 \to K_{n-1} \xrightarrow{\alpha} L_{n-1} \xrightarrow{\beta} M_{n-1} \to 0$$

is exact, it follows that $\partial(l) \in \operatorname{im} \alpha$

$\Rightarrow \exists k \in K_{n-1} : \alpha(k) = \partial l$

Then $\alpha \partial'(k) = \partial \alpha(k) = \partial\partial(l) = 0$

α inj $\Rightarrow \partial'(k) = 0$, i.e., $k \in Z_{n-1}(K)$.

Let $l_1 \in L_n$ be another element s.t. $\beta(l_1) = m$.
Then there is a unique $k_1 \in K_{n-1}$ with $\alpha(k_1) = \partial(l_1)$.
Now,

$$\beta(l - l_1) = \beta(l) - \beta(l_1) = m - m = 0.$$

Since $\ker \beta = \operatorname{im} \alpha$, there is $\tilde{k} \in K_n : \alpha(\tilde{k}) = l - l_1$.
Since

$$\alpha \partial'(\tilde{k}) = \partial\alpha(\tilde{k}) = \partial(l - l_1) = \partial(l) - \partial(l_1)$$

$$= \alpha(k) - \alpha(k_1) = \alpha(k - k_1)$$

and α is an injection, it follows that

$$\partial'(\tilde{k}) = k - k_1.$$

Thus $k - k_1 \in B_{n-1}(K)$. $\Rightarrow [k] = [k_1] \in H_{n-1}(K)$.

We obtain a well defined function

$$\hat{\Delta}_n : Z_n(M) \to H_{n-1}(K)$$
$$m \mapsto [k]$$
as above

Here $\alpha(k) = \partial l$, where $\beta(l) = m$.