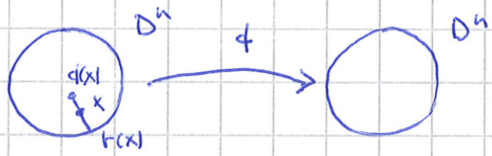
Homology

__Example__ (Brouwer) Let $f : D^n \to D^n$ be a continuous map, $n \geq 1$. Then there is $x \in D^n$ with $f(x) = x$.

Case $n=1$ is easy and we already proved the case $n=2$. Let's imitate the proof for $n=2$ in the general case. Assume there is $f : D^n \to D^n$, $f(x) \neq x$ for all x. Let l_x be the half line starting at $f(x)$ and going through x. Let $r(x)$ be the unique point where l_x intersects S^{n-1} (= the boundary of D^n).



The mapping $r : D^n \to S^{n-1} \subset D^n$, $x \mapsto r(x)$, is continuous and $r(x) = x$ for all $x \in S^{n-1}$. Thus r is a retraction. Let $i : S^{n-1} \to D^n$, $x \mapsto x$, be the inclusion. Then $r \circ i : S^{n-1} \to S^{n-1}$ is the identity map id of S^{n-1}.

$n=1$: $S^0 = \{-1, 1\}$, $D^1 = [-1, 1]$

There is no retraction (no continuous surjection) $D^1 \to S^0$, since D^1 is connected but S^0 is not.

$n=2$:

$$\begin{array}{ccccc} \pi_1(S^1, 1) & \xrightarrow{i_*} & \pi_1(D^2, 1) & \xrightarrow{r_*} & \pi_1(S^1, 1) \\ \cong & & \cong & & \cong \\ \mathbb{Z} & & \{1\} & & \mathbb{Z} \end{array}$$

If $r \circ i = \mathrm{id}$, then $r_* \circ i_* = (r \circ i)_* = \mathrm{id}_* = \mathrm{id} : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$, which is impossible. Thus there is no retraction $D^2 \to S^1$.

$n > 2$: $\pi_1(S^{n-1}, 1) \cong \{1\}$, $\pi_1(D^n, 1) \cong \{1\}$

Thus the argument like in the case $n=2$ does not work.

The case $n > 2$ can be proved by using homology:

Let X = topological space: There are abelian groups $H_0(X), H_1(X), \dots$ called the homology groups of X. A continuous map $h: X \to Y$ induces a group homomorphism $h_*: H_n(X) \to H_n(Y)$, for all $n \geq 0$. If $g: Y \to Z$ is continuous, then $(g \circ h)_* = g_* \circ h_*$.

$$X \xrightarrow{h} Y \xrightarrow{g} Z$$

$$\begin{tikzcd} H_n(X) \arrow[r, "h^*"] \arrow[rr, bend right, "g_* \circ h_*"'] & H_n(Y) \arrow[r, "g^*"] & H_n(Z) \end{tikzcd}$$

Assume $r: D^n \to S^{n-1}$ is a retraction. The commutative diagram

$$\begin{tikzcd} D^n \arrow[r, "r"] & S^{n-1} \\ S^{n-1} \arrow[u, "i"] \arrow[ur, "id"'] & \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} H_{n-1}(D^n) \arrow[r, "r_*"] & H_{n-1}(S^n) \\ H_{n-1}(S^n) \arrow[u, "i_*"] \arrow[ur, "(id)_* = id"'] & \end{tikzcd}$$

(also for H_m, $m \geq 0$, not only for H_{n-1})

Calculate homology groups:

$H_m(D^n) \cong \{1\}$ $\forall m > 0$ $\qquad H_m(S^{n-1}) = \begin{cases} \mathbb{Z}, & m = n-1 \\ 0, & m \neq n-1 \end{cases}$ $\quad (m > 0)$

Therefore, for $m = n-1$, we obtain:

$$\begin{tikzcd} \{1\} \arrow[r, "r_*"] & \mathbb{Z} \\ \mathbb{Z} \arrow[u, "i_*"] \arrow[ur, "(id)_* = id"'] & \end{tikzcd}$$

This is impossible. Thus there is no retraction $D^n \to S^{n-1}$. "$\square$"

11. Eilenberg - Steenrod axioms

Let X = topological space, $A \subset X$. Then (X, A) is called a topological pair. A continuous function $f: (X, A) \to (Y, B)$ means a continuous function $f: X \to Y$ with $f(A) \subset B$. We write X for $(X, \emptyset)$.

A <u>homology theory</u> H defined for all topological pairs (X, A) and for all continuous functions $f: (X, A) \to (Y, B)$ consists of the following:

For every topological pair (X, A) and for every $n \in \mathbb{Z}_+ \cup \{0\}$ there is an abelian group $H_n(X, A)$. Every continuous function $f: (X, A) \to (Y, B)$ induces a group homomorphism

$$f_*: H_n(X, A) \to H_n(Y, B), \text{ for all } n \in \mathbb{Z}_+ \cup \{0\}$$

For every (X, A) and for every $n \in \mathbb{Z}_+$ there is a group homomorphism

$$\partial: H_n(X, A) \to H_{n-1}(A)$$

such that the following axioms (called Eilenberg-Steenrod axioms) hold:

A1. The identity map $id:(X,A)\to(X,A)$ induces $id_* = id : H_n(X,A)\to H_n(X,A)$, for all $n\geq 0$.

A2. If $f:(X,A)\to(Y,B)$ and $g:(Y,B)\to(Z,C)$ are continuous, then $(g\circ f)_* = g_*\circ f_* : H_n(X,A)\to H_n(Z,C)$, for all $n\geq 0$.

A3. If $f:(X,A)\to(Y,B)$ is continuous, then the diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f|A)_*} \\ H_n(Y,B) & \xrightarrow[\partial]{} & H_{n-1}(B) \end{array}$$

commutes, for all $n\geq 1$.

A4. If (X,A) is a topological pair and $i:A\hookrightarrow X$ is the inclusion and let $j: X=(X,\phi)\to(X,A)$ be the inclusion, then the sequence

$$\cdots\xrightarrow{j_*} H_{n+1}(X,A)\xrightarrow{\partial} H_n(A)\xrightarrow{i_*} H_n(X)\xrightarrow{j_*} H_n(X,A)\xrightarrow{\partial} H_{n-1}(A)\xrightarrow{i_*}\cdots$$

is exact. (Exactness axiom)

(A sequence of abelian groups and homomorphisms

$$\cdots\to S_{n+1}\xrightarrow{f_{n+1}} S_n\xrightarrow{f_n} S_{n-1}\to\cdots$$

is called exact if $\operatorname{im} f_{n+1} = \ker f_n$ for all n.)

A5. If $f,g:(X,A)\to(Y,B)$ are homotopic (i.e., if there is a homotopy $F:(X\times I, A\times I)\to(Y,B)$ with $F_0=f$ and $F_1=g$), then $f_*=g_*: H_n(X,A)\to H_n(Y,B)$, for all $n\geq 0$. (Homotopy axiom)

A6. For every topological pair (X, A) and for every open subset U of X with $\overline{U} \subset \operatorname{int} A$ (also denoted by $\mathring{A}$), the inclusion $i : (X-U, A-U) \to (X, A)$ induces group isomorphisms

$$i_* : H_n(X-U, A-U) \to H_n(X, A), \text{ for all } n \geq 0.$$

(Excision axiom, excision = typistys in Finnish)

A7. If X is a one-point space, then $H_n(X) = 0$ for all $n > 0$. (The group $H_0(X, \emptyset)$ is called the coefficient group.) (Dimension axiom)

12. Singular homology theory

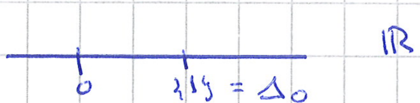
Theorem. There exists a homology theory H defined for all topological pairs (X, A) and for all continuous maps $f : (X, A) \to (Y, B)$ satisfying all 7 Eilenberg - Steenrod axioms.

We will prove this theorem by constructing the singular homology theory (S. Eilenberg ~ 1947).

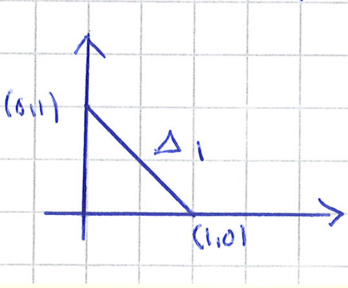
Recall: The standard n-simplex $(n \geq 0)$ is

$$\Delta_n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\ \forall i \in \{0, \dots, n\}\}.$$

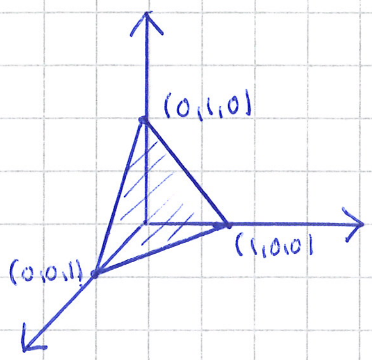
$n = 0$: $\Delta_0 = \{t_0 \in \mathbb{R} \mid t_0 = 1\} = \{1\}$



$n = 1$: $\Delta_1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1,\ t_0 \geq 0,\ t_1 \geq 0\}$



$n=2$: $\Delta_2 = \{(t_0,t_1,t_2) \in \mathbb{R}^3 \mid t_0+t_1+t_2 = 1,\ t_0 \geq 0,\ t_1 \geq 0,\ t_2 \geq 0\}$



Thus, if $p_i = (0,\dots,0,\overset{i}{1},0,\dots,0)$, then Δ_n is the convex hull of $p_0,\dots,p_n$.

Define $e^j_n = e^j : \Delta_{n-1} \to \Delta_n$, $0 \leq j \leq n$, $n \geq 1$,

$$e^j(\underset{0}{t_0},\dots,\underset{n-1}{t_{n-1}}) = (\underset{0}{t_0},\dots,\underset{j-1}{t_{j-1}},\underset{j}{0},\underset{j+1}{t_j},\dots,\underset{n}{t_{n-1}})$$

$n=1$: $e^0 : \Delta_0 \to \Delta_1$ $\quad e^0(1) = (0,1)$
$\quad\ \ e^1 : \Delta_0 \to \Delta_1$ $\quad e^1(1) = (1,0)$

$n=2$: $e^0 : \Delta_1 \to \Delta_2$ $\quad e^0(t_0,t_1) = (0,t_0,t_1)$
$\quad\ \ e^1 : \Delta_1 \to \Delta_2$ $\quad e^1(t_0,t_1) = (t_0,0,t_1)$
$\quad\ \ e^2 : \Delta_1 \to \Delta_2$ $\quad e^2(t_0,t_1) = (t_0,t_1,0)$

Lemma 12.1. $e^j_n \circ e^k_{n-1} = e^k_n \circ e^{j-1}_{n-1} : \Delta_{n-2} \to \Delta_n$, if $0 \leq k < j \leq n$, $n \geq 2$.

proof. Assume $k = j-1$. Then

$$e^j_n \circ e^k_{n-1}(t_0,\dots,t_{n-2}) = e^j_n(\underset{0}{t_0},\dots,\underset{k-1}{t_{k-1}},\underset{k}{0},\underset{k+1}{t_k},\dots,\underset{n-1}{t_{n-2}})$$

$$= (t_0,\dots,t_{k-1},\underset{k}{0},\underset{k+1}{0},t_k,\dots,t_{n-2})$$

and

$$e_n^k \circ e_{n-1}^{j-1}(t_0,\dots,t_{n-2}) = e_n^k(\underset{0}{t_0},\dots,\underset{j-2}{t_{j-2}},\underset{\substack{k\\ \shortparallel \\ j-1}}{0},\underset{j}{t_{j-1}},\dots,\underset{n-1}{t_{n-2}})$$

$$= (t_0,\dots,t_{j-2},0,0,t_{j-1},\dots,t_{n-2}).$$

Since $j-2=k-1$, it follows that $e_n^j \circ e_{n-1}^k(t_0,\dots,t_{n-2}) = e_n^k \circ e_{n-1}^{j-1}(t_0,\dots,t_{n-2})$ $\forall\, (t_0,\dots,t_{n-2}) \in \Delta_{n-2}$.

$$\therefore\ e_n^j \circ e_{n-1}^k = e_n^k \circ e_{n-1}^{j-1}$$

$k < j-1$: homework. $\square$

The map e_n^j is called the j^{th} face map of the simplex Δ_n.

Definition 12.2. Let X be a topological space. A continuous map $T: \Delta_n \to X$ is called a singular n-simplex of X.

Define

$$Map(\Delta_n; X)$$

to be the set of all continuous functions $\Delta_n \to X$, i.e, the set of all singular n-simplexes of X.

Let $T \in Map(\Delta_n; X)$. Let $\mathbb{Z}_T \cong \mathbb{Z}$ be the infinite cyclic group generated by T. Then the elements in $\mathbb{Z}_T$ are of the form mT, where $m \in \mathbb{Z}$.
Define

$$S_n(X) = \sum_{T \in Map(\Delta_n; X)} \oplus\, \mathbb{Z}_T, \quad n \geq 0,$$

to be the free abelian group with basis all singular n-simplexes in X. Therefore, the elements of $S_n(X)$ are of the form $\sum_{T \in Map(\Delta_n; X)} n_T T$, where $n_T \in \mathbb{Z}$ and $n_T = 0$ except for finitely many T (i.e. $\sum_{i=1}^{k} n_i T_i = n_1 T_1 + \dots + n_k T_k$).

The elements of $S_n(X)$ are called (singular) <u>n-chains</u> in X.

Let $T: \Delta_n \to X$, $(t_0, \dots, t_n) \mapsto T(t_0, \dots, t_n)$.

The <u>j-face</u> of T, $0 \le j \le n$, is

$$T^{(j)} = T \circ e^j : \Delta_{n-1} \to X, \ (t_0, \dots, t_{n-1}) \mapsto T(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n+1}).$$

Define $S_{-1}(X)$ to be $\{0\}$. We define the <u>boundary homomorphism</u>

$$\partial : S_n(X) \to S_{n-1}(X)$$

as follows: If $T \in Map(\Delta_n, X)$, we define

$$\partial T = \sum_{i=0}^{n} (-1)^i T^{(i)}$$

and $\partial : S_n(X) \to S_{n-1}(X)$ is obtained by requiring ∂ to be linear (i.e., $\partial(n_1 T_1 + \dots + n_k T_k) = n_1 \partial T_1 + \dots + n_k \partial T_k$), for $n > 0$. If $n = 0$, then $\partial : S_0(X) \to \{0\}$, is the constant map 0.

We have constructed a sequence of free abelian groups and homomorphisms

$$\to S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \to S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0,$$

called the <u>singular complex</u> of X and denoted by $(S_*(X), \partial)$ or by $S_*(X)$.