Definition 9.5. Let $\alpha : I \to S^1$ be a closed path with $\alpha(0) = \alpha(1) = 1$. Let $\alpha' : I \to \mathbb{R}$ be the unique lift of α with $\alpha'(0) = 0$. We call $\alpha'(1) \in \mathbb{Z}$ the degree of α and denote it ~~by~~ $\deg \alpha$.

Notice: $p(\alpha'(1)) = \alpha(1) = 1 \Rightarrow \alpha'(1) \in \ker p = \mathbb{Z}$.

If $\alpha \simeq \beta$ rel $\dot{I}$, then $\alpha' \simeq \beta'$ rel $\dot{I}$. Thus

$$\deg \alpha = \alpha'(1) = \beta'(1) = \deg \beta.$$

Thus we obtain a function

$$\deg : \pi_1(S^1, 1) \to \mathbb{Z}, \quad [\alpha] \mapsto \deg \alpha$$

Proposition 9.6. The function

$$\deg : \pi_1(S^1, 1) \to \mathbb{Z}$$

is a group isomorphism.

proof: Three parts: 1) deg is an injection
2) deg is a surjection
3) deg is a group homomorphism.

1) Assume $\deg[\alpha] = \deg[\beta]$. Let α' and β' be the lifts of α and β, respectively. Then $\alpha'(1) = \beta'(1)$ and $\alpha'(0) = \beta'(0) = 0$. Let

$$F : I \times I \to \mathbb{R}, \quad (s,t) \mapsto (1-t)\alpha'(s) + t\beta'(s).$$

Then $F : \alpha' \simeq \beta'$ rel $\dot{I}$. $\Rightarrow pF : \alpha \simeq \beta$ rel $\dot{I}$

$\Rightarrow [\alpha] = [\beta]$

$\therefore$ deg is an injection

2) Let $n \in \mathbb{Z}$. Let $\alpha_n : I \to S^1$, $s \mapsto e^{i2\pi ns}$.
The lift of α_n is $\alpha_n' : I \to \mathbb{R}$, $s \mapsto ns$.
Then $\alpha_n'(1) = n$, which implies that $\deg[\alpha_n] = n$.

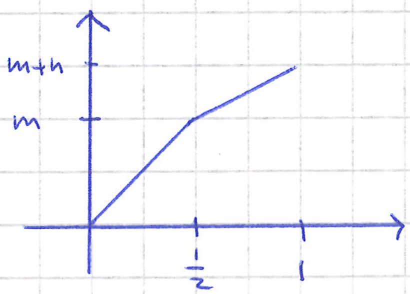
$\therefore$ deg is a surjection.

3) It remains to show that deg is a group homomorphism. 1 and 2 $\Rightarrow \pi_1(S^1, 1) = \{[\alpha_n] \mid n \in \mathbb{Z}\}$. Since $\deg[\alpha_n] = n$, we will show that $\deg[\alpha_m * \alpha_n] = m+n$. Now,

$$(\alpha_m * \alpha_n)(s) = \begin{cases} e^{i2\pi m 2s}, & 0 \le s \le \frac{1}{2} \\ e^{i2\pi n(2s-1)}, & \frac{1}{2} \le s \le 1 \end{cases}.$$

The path $\alpha_m * \alpha_n$ has a lift

$$(\alpha_m * \alpha_n)'(s) = \begin{cases} 2sm, & 0 \le s \le \frac{1}{2} \\ (2s-1)n + m, & \frac{1}{2} \le s \le 1 \end{cases}$$



and $(\alpha_m * \alpha_n)'(1) = m+n$.

Thus $\deg[\alpha_m * \alpha_n] = (\alpha_m * \alpha_n)'(1) = m+n$.

$\square$

<u>Corollary 9.7.</u> S^1 is not a retract of D^2.

<u>proof.</u> Counter assumption: Assume there is a retraction $r : D^2 \to S^1$. Let $i : S \hookrightarrow D^2$ be the canonical inclusion. Then both r and i are based maps w.r.t the basepoint $1 \in S^1$. Since $r \circ i = \mathrm{id}_{S^1}$,

$$r_* \circ i_* = (r \circ i)_* = \mathrm{id}_* = \mathrm{id} : \pi_1(S^1, 1) \to \pi_1(S^1, 1).$$

But

$$\mathbb{Z} \cong \pi_1(S^1, 1) \xrightarrow{i_*} \pi_1(D^2, 1) \xrightarrow{r_*} \pi_1(S^1, 1) \cong \mathbb{Z}.$$

Since $\pi_1(D^2, 1) = 0$, $r_* \circ i_*$ cannot be an isomorphism. $\square$

More generally we may conclude the following: Assume $r: X \to A$ is a based retraction (i.e., a based map that also is a retraction). Then:

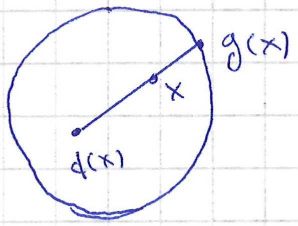
$i_*: \pi_1(A, a_0) \longrightarrow \pi_1(X, a_0)$ is an injection

$r_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$ is a surjection.

Corollary 9.8. (Brouwer fixed point theorem)

Every continuous map $f: D^2 \to D^2$ has a fixed point (i.e., $\exists\ x_0 \in D^2$ s.t. $f(x_0) = x_0$).

proof. Counterassumption: $\exists$ a continuous map $f: D^2 \to D^2$ s.t. $f(x) \neq x$ $\forall x \in D^2$. Define $g: D2 \to S^1$: $g(x)$ is the point where the half line starting at $f(x)$ and going through x meets S^1. Then g is continuous (check this!). Clearly, $g: D^2 \to S^1$ is a retraction. Contradiction. □



Corollary 9.9 S^1 is not simply connected.

Corollary 9.10. Closed paths f and g in S^1 at 1 are homotopic rel $\dot{I}$ if and only if $\deg f = \deg g$.

proof. $\deg: \pi_1(S^1, 1) \to \mathbb{Z}$ is well defined $\Rightarrow$ $\deg f = \deg g$ if $f \simeq g$ rel $\dot{I}$. Since $\deg$ is injective, $\deg f = \deg g$ implies that $[f] = [g]$. □

Theorem (9.11) (Fundamental Theorem of Algebra)

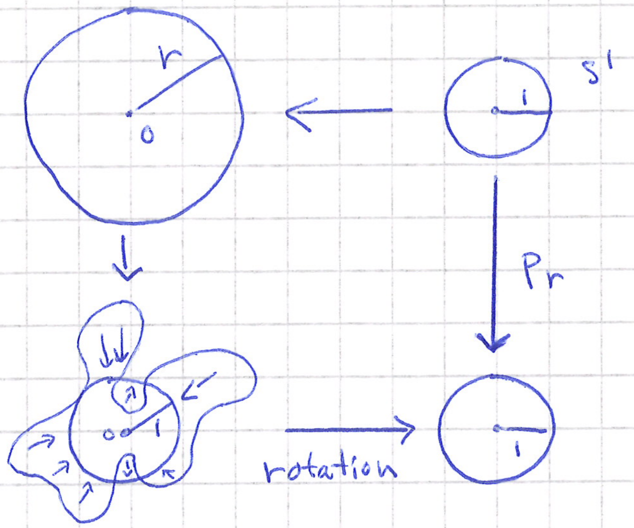
Every complex polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ ($a_i \in \mathbb{C}$ $\forall i$, $a_n \neq 0$, $n > 0$) has a root $z_0 \in \mathbb{C}$.

proof. We may assume that $a_n = 1$ (if $a_n \neq 1$, we may divide p by a_n and obtain a polynomial that has the same roots as p). Assume that $p(z) \neq 0$, for all $z \in \mathbb{C}$. Define

$$P : S^1 \times [0, \infty) \to S^1, \quad P(z, r) = \frac{p(rz)}{|p(rz)|} \cdot \frac{|p(r)|}{p(r)}.$$

Then P is continuous. Let

$$p_r : S^1 \to S^1, \quad p_r(z) = P(z, r).$$



Claim 1: $[p_r] \in \pi_1(S^1, 1)$ does not depend on r.

Claim 2: $\deg p_0 = 0$.

Claim 3: $\deg p_1 = n$.

$\Rightarrow$ contradiction, since if $[p_0] = [p_1]$ then $\deg p_0 = \deg p_1$.

proof of Claim 1: Let $r_1, r_2 \geq 0$. Let

$$F : S^1 \times I \to S^1, \quad F(z, t) = P(z, (1-t) r_1 + t r_2).$$

Then $F : p_{r_1} \simeq p_{r_2}$ rel 1.

proof of Claim 2: $p_0(z) = \dfrac{p(0 \cdot z)}{|p(0 \cdot z)|} \cdot \dfrac{|p(0)|}{p(0)} = 1$ for all $z \in S^1$
$\Rightarrow [p_0] = \{1\} \Rightarrow \deg p_0 = 0$.

<u>proof of Claim 3.</u> Let $t > 0$. Then

$$p\left(\frac{z}{t}\right) = \frac{1}{t^n}\left(a_n z^n + a_{n-1} t z^{n-1} + \dots + a_1 t^{n-1} z + a_0 t^n\right).$$

$$\Rightarrow \quad \frac{p(z/t)}{|p(z/t)|} = \frac{a_n z^n + a_{n-1} t z^{n-1} + \dots + a_1 t^{n-1} z + a_0 t^n}{|a_n z^n + a_{n-1} t z^{n-1} + \dots + a_1 t^{n-1} z + a_0 t^n|}.$$

As a function of (z,t) this is continuous also at points $(z,0)$, where the right side obtains the value

$$\frac{a_n z^n}{|a_n z^n|} = \frac{a_n}{|a_n|} \cdot \underbrace{\frac{1}{|z^n|}}_{=1 \text{ since } z^n \in S^1} z^n = \frac{a_n}{|a_n|} z^n \underset{\uparrow a_n = 1}{=} z^n$$

Thus the function

$$F : S^1 \times [0,\infty) \to S^1, \ (z,t) \mapsto \begin{cases} z^n, & \text{if } t = 0 \\ P(z, \frac{1}{t}), & \text{if } 0 < t \le 1 \end{cases}$$

is continuous. Let $\mu_n : S^1 \to S^1$, $z \mapsto z^n$. Then:

$t = 0$: $F(z,0) = z^n = \mu_n(z) \quad \forall z \in S^1$.

$t = 1$: $F(z,1) = P(z,1) = p_1(z) \quad \forall z \in S^1$.

$z = 1$: $F(1,t) = \begin{cases} 1^n = 1, & \text{if } t = 0 \\ P(1, 1/t) = \dfrac{p(1 \cdot 1/t)}{|p(1 \cdot 1/t)|} \cdot \dfrac{|p(1/t)|}{p(1/t)} = 1, & \text{if } 0 < t \le 1. \end{cases}$

Thus $F : \mu_n \simeq p_1$ rel 1. Hence $\deg p_1 = \deg \mu_n$.
Let $\beta : I \to S^1$, $t \mapsto e^{i2\pi t}$, and let

$$\alpha_n : I \to S^1, \ t \mapsto (\mu_n \circ \beta)(t) = (e^{i2\pi t})^n = e^{in2\pi t}.$$

$$\begin{tikzcd} I \arrow[r, "\alpha'_n"] \arrow[d, "\beta"'] \arrow[dr, "\alpha_n"] & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[r, "\mu_n"'] & S^1 \end{tikzcd}$$

Then $\alpha_n(0) = \alpha_n(1) = 1$ and α_n has a lift $\alpha'_n : I \to \mathbb{R}$, $\alpha'_n(t) = nt$. Then $\alpha'_n(0) = 0$, $\alpha'_n(1) = n$.

Thus $\deg p_1 = \deg \mu_n = \deg \alpha_n = \alpha'_n(1) = n$. $\square$

10. Seifert - van Kampen theorem

Recall the following:

Theorem 10.1 (Lebesgue number theorem)

Let X be a compact metric space and let $\mathcal{U}$ be an open cover of X. Then there exists $\lambda > 0$ such that every open ball of radius less than λ lies in some element of $\mathcal{U}$. $\square$

Theorem 10.2 (Seifert - van Kampen)

Let X be a topological space. Assume $X = X_1 \cup X_2$, where X_1 and X_2 are open, simply connected subsets of X. Assume $\emptyset \neq X_0 = X_1 \cap X_2$ is path connected. Then $\pi_1(X, x_0) = \{1\}$ for all $x_0 \in X_0$. $\leftarrow$ (In fact, $\pi_1(X, x_0) = \{1\}$ $\forall x_0 \in X$, since X is path. conn.)

proof. Let $x_0 \in X_0$ and let $\alpha : I \to X$ be a closed path at x_0. Then $\{\alpha^{-1}(X_1), \alpha^{-1}(X_2)\}$ is an open cover of I. Since I is compact, it follows from Theorem 10.1 that there exists $\lambda > 0$ such that every subinterval J of I, whose length is less than λ, lies in $\alpha^{-1}(X_1)$ or in $\alpha^{-1}(X_2)$.

Let $n \in \mathbb{N}$, $\frac{1}{n} < \lambda$. Then

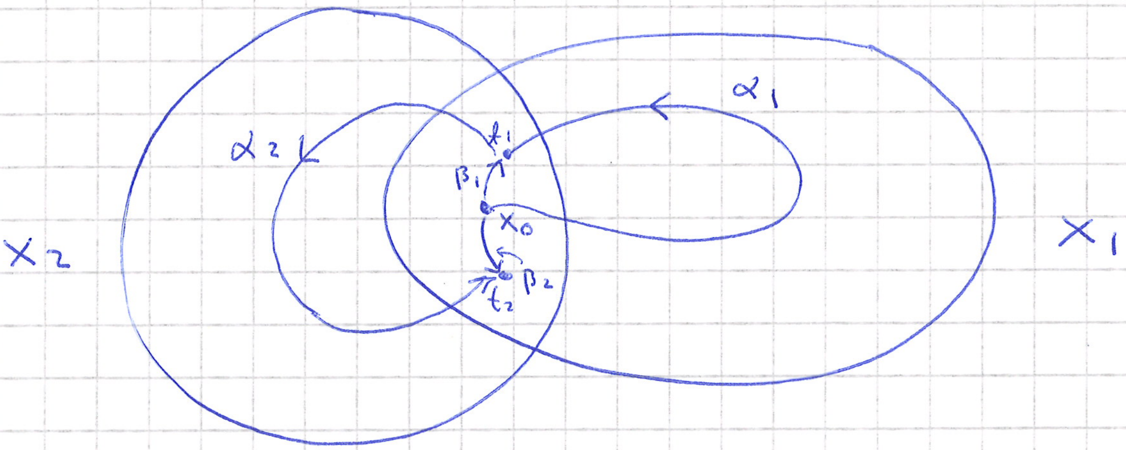
$$0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < 1$$

and $\alpha[\frac{k-1}{n}, \frac{k}{n}] \subset X_1$ or $\alpha[\frac{k-1}{n}, \frac{k}{n}] \subset X_2$ $\forall k \in \{1, \dots, n\}$.
By deleting some of the points $\frac{k}{n}$ if necessary, we obtain $t_i \in [0,1]$:

$$0 = t_0 < t_1 < \dots < t_m = 1$$

such that $\alpha[t_{i-1}, t_i]$ and $\alpha[t_i, t_{i+1}]$ lie in different sets X_1, X_2. Then $\alpha(t_i) \in X_1 \cap X_2 = X_0$ $\forall i$.

Since X_0 is path connected there is a path $\beta_i : I \to X_0$ from x_0 to $\alpha(t_i)$, for every i. Let $\alpha_i : I \to X$ be the path $\alpha|[t_{i-1}, t_i]$ parametrized so that it is defined on the unit interval I.



Then $\alpha \simeq \alpha_1 * \alpha_2 * \dots * \alpha_m$

$$\simeq (\alpha_1 * \beta_1^{-1}) * (\beta_1 * \alpha_2 * \beta_2^{-1}) * \dots * (\beta_{m-1} * \alpha_m) \text{ rel } \dot{I}. \qquad (*)$$

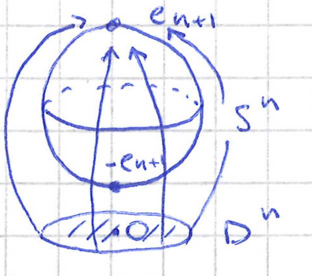
closed paths at x_0, each of these paths lie in X_1 or in X_2

Since X_1, X_2 are simply connected, each path $\beta_i * \alpha_{i+1} * \beta_{i+1}^{-1}$ is null homotopic rel $\dot{I}$. Thus also α is null homotopic rel $\dot{I}$. □

Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ and $\mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.

<u>Lemma 10.3.</u> There is a homeomorphism $S^n - \{e_{n+1}\} \to \mathbb{R}^n$, for every $n \geq 1$, where $e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Also, there is a homeom $D^n / S^{n-1} \to S^n$.

<u>Proof.</u>



Let $\pi : D^n \to S^n$, $\pi(y) = (\underbrace{2\sqrt{1-\|y\|^2}\, y}_{\in \mathbb{R}^n}, \underbrace{2\|y\|^2 - 1}_{\in \mathbb{R}})$.

Then π is welldefined, since $\|\pi(y)\| = 1$ for all $y \in D^n$.
Also, $\pi^{-1}(\{e_{n+1}\}) = S^{n-1}$.
The restriction of π to $\mathring{D}^n$ is a homeomorphism $\mathring{D}^n \to S^n - \{e_{n+1}\}$. The inverse is given by
$\rho : S^n - \{e_{n+1}\} \to \mathring{D}^n$, $(z, t) \mapsto \frac{z}{\sqrt{2(1-t)}}$.

The map π induces a continuous bijection

$$\bar{\pi} : D^n / S^{n-1} \longrightarrow S^n,$$

$$\begin{tikzcd}
D^n \arrow[d, "p"'] \arrow[dr, "\pi"] & \\
D^n / S^{n-1} \arrow[r, "\bar{\pi}"'] & S^n
\end{tikzcd}$$

$p =$ quotient map

Since D^n / S^{n-1} is compact (D^n compact, p continuous surjection $\Rightarrow D^n/S^{n-1}$ compact) and S^n is Hausdorff, it follows that $\bar{\pi}$ is a homeomorphism.

The map $\pi' : \mathring{D}^n \to \mathbb{R}^n$, $y \mapsto \frac{y}{1 - \|y\|}$,

is a homeomorphism with the inverse

$$\rho' : \mathbb{R}^n \to \mathring{D}^n, \quad z \mapsto \frac{z}{1 + \|z\|}.$$

Then the composed map

$$S^n - \{e_{n+1}\} \xrightarrow{\rho} \mathring{D}^n \xrightarrow{\pi'} \mathbb{R}^n$$

is a homeomorphism. $\square$

<u>Notice</u>: e_{n+1} could be replaced by any $x_0 \in S^n$.

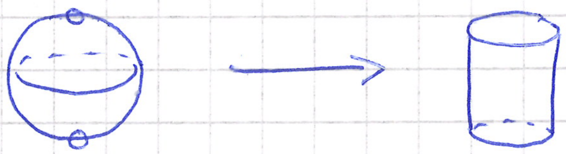
Theorem 10.4. The sphere S^n is simply connected for all $n \geq 2$.

proof. Let $U_1 = S^n - \{e_{n+1}\}$, $U_2 = S^n - \{-e_{n+1}\}$. Then $S^n = U_1 \cup U_2$.

Lemma 10.3 $\Rightarrow$ $U_1 \cong \mathbb{R}^n \cong U_2$ ($\cong$ means homeomorphic).

Now, $\mathbb{R}^n$ is path connected for all n and $U_1 \cap U_2 \neq \emptyset$ for $n \geq 1$. Thus S^n is path connected for $n \geq 1$. $\mathbb{R}^n$ simply connected $\Rightarrow$ U_1, U_2 simply connected.

Also, $f: U_1 \cap U_2 \to S^{n-1} \times (-1,1)$, $(z,t) \mapsto \left(\frac{z}{\|z\|}, t\right)$, is a homeomorphism:



The inverse is $f^{-1}: S^{n-1} \times (-1,1) \to U_1 \cap U_2$, $(z,t) \mapsto (\sqrt{1-t^2}\, z, t)$. Thus $U_1 \cap U_2$ is pathconnected for $n \geq 2$.

Seifert-van Kampen Theorem $\Rightarrow$ $\pi_1(S^n, x_0) = \{1\}$, for all $x_0 \in U_1 \cap U_2 = S^n - \{\pm e_{n+1}\}$. Since S^n is path connected, it follows from Theorem 8.3 that also $\pi_1(S^n, e_{n+1}) = \{1\} = \pi_1(S^n, -e_{n+1})$, for $n \geq 2$. $\square$

Corollary 10.5. S^1 and S^n do not have the same homotopy type for $n > 1$.

proof. By Corollary 8.8, topol. spaces that have the same homotopy type, have isomorphic fundamental groups. $\square$

Corollary 10.6. $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic, for $n \geq 3$.

proof. Counter assumption: Assume there is a homeom. $f: \mathbb{R}^2 \to \mathbb{R}^n$. If $f(0) = x_0$, then

$$f': \mathbb{R}^2 \to \mathbb{R}^n, \quad x \mapsto f(x) - x_0,$$

is a homeom. satisfying $f'(0) = 0$. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection with $A(f'(e_1)) = e_1$ (f' bijection, $f'(0) = 0 \Rightarrow f'(e_1) \neq 0$). Then

$$g = A \circ f' : \mathbb{R}^2 \to \mathbb{R}^n$$

is a homeomorphism and

$$g(0) = (A \circ f')(0) = A(f'(0)) = A(0) = 0, \quad g(e_1) = A(f'(e_1)) = e_1.$$

The restriction of $A \circ f'$,

$$h: \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^n - \{0\}, \quad x \mapsto g(x),$$

is a homeomorphism and $h(e_1) = e_1$. Let $i: S^1 \hookrightarrow \mathbb{R}^2 - \{0\}$ be the inclusion and let

$$r: \mathbb{R}^n - \{0\} \to S^{n-1}, \quad x \mapsto \frac{x}{\|x\|}.$$

Then i and r are homotopy equivalences. Thus the composed map

$$S^1 \xhookrightarrow{i} \mathbb{R}^2 - \{0\} \xrightarrow{h} \mathbb{R}^n - \{0\} \xrightarrow{r} S^{n-1}$$

is a homotopy equivalence. Contradiction. $\square$

Definition 10.7. A **topological group** G is a group equipped with a topology such that sets consisting of one point are closed and

1) the multiplication map $\mu : G\times G \to G$, $(x,y)\mapsto xy$, is continuous if $G\times G$ has the product topology,

2) the inversion map $i : G\to G$, $x\mapsto x^{-1}$, is continuous.

Example $(\mathbb{R},+)$ and $(\mathbb{Z},+)$ are topological groups. S^1 equipped with complex multiplication is a topological group. Any group equipped with the discrete topology is a topological group.

Definition 10.8. Let (X,x_0) be a pointed space. Assume there is a pointed map $m : (X\times X,(x_0,x_0)) \to (X,x_0)$ such that the pointed maps $m(\ ,x_0)$ and $m(x_0,\ )$ are homotopic to 1_X rel $\{x_0\}$. Then (X,x_0) is called an H-space (after H. Hopf).

Example Every topological group X with identity x_0 is an H-space.

Let (X,x_0) be an H-space. Let $k : X\to X$, $x\mapsto x_0$ and let $1_X : X\to X$, $x\mapsto x$. Then $m(x_0,\ ) = m\circ(k,1_X)$, where $(k,1_X) : X\to X\times X$, $x\mapsto (x_0,x)$. Similarly, $m(\ ,x_0) = m\circ(1_X,k)$. Thus both $m\circ(k,1_X)$ and $m\circ(1_X,k)$ are homotopic to 1_X rel $\{x_0\}$.

Recall: Let G and H be groups with identity elements e and e', respectively. Let $x\in G$ and let $y\in H$. Then

$$(x,e')(e,y) = (x,y) = (e,y)(x,e').$$

Theorem 10.9. Let (X, x_0) be an H-space. Then $\pi_1(X, x_0)$ is abelian.

proof. By Theorem 8.4, the map

$$\theta : \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X \times X, (x_0, x_0)), \ ([f],[g]) \mapsto [(f,g)],$$

is a group isomorphism. (Here $(f,g) : I \to X \times X$, $t \mapsto (f(t), g(t))$.) Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$[g] = (m \circ (k, 1_X))_* [g] \qquad (\text{definition of H-space})$$

$$= m_* (k, 1_X)_* [g]$$

$$= m_* [(k, 1_X) \circ g]$$

$$= m_* [(kg, g)]$$

$$= m_* \theta([kg], [g]) \qquad (\text{definition of } \theta)$$

$$= m_* \theta(e, [g]),$$

where $e = [k]$ is the identity element of $\pi_1(X, x_0)$. Similarly,

$$[f] = m_* \theta([f], e),$$

since $m \circ (1_X, k) \simeq 1_X \ \mathrm{rel}\{x_0\}$. The composition

$$m_* \theta : \pi_1(X, x_0) \times \pi_1(X, x_0) \xrightarrow{\theta} \pi_1(X \times X, (x_0, x_0)) \xrightarrow{m_*} \pi_1(X, x_0)$$

is a homomorphism. Therefore,

$$m_* \theta([f], [g]) = m_* \theta((e, [g])([f], e))$$

$$= m_* \theta((e, [g]))\, m_* \theta(([f], e))$$

$$= [g][f].$$

By writing $([f],[g]) = ([f],e)(e,[g])$, one sees that $m_* \theta([f],[g]) = [f][g]$. Then $[g][f] = [f][g]$ and it follows that $\pi_1(X,x_0)$ is abelian.

$\square$

Corollary 10.10. Let G be a topological group. Then $\pi_1(G,e)$ is abelian.

Frank Adams (Hopf invariant one theorem):

S^0, S^1, S^3, S^7 are the only spheres that are H-spaces. (S^0, S^1, S^3 are Lie groups, S^7 is not a group)

Comments about topological groups:

Lemma 10.11. Topological groups are Hausdorff spaces.

proof Recall that a topological space X is Hausdorff if and only if its diagonal $\Delta X = \{(x,x) | x \in X\}$ is closed in $X \times X$. Let G be a topological group. The map $f: G \times G \to G$, $(g,h) \mapsto gh^{-1}$, is continuous. Let e be the identity element of G. Since $\{e\}$ is closed in G, it follows that

$$\Delta = \{(g,g) | g \in G\} = f^{-1}(e)$$

is closed in $G \times G$. Then G is Hausdorff. $\square$

Let G be a topological group and let $h \in G$. Then h can be considered as a homeomorphism $G \to G$, $g \mapsto hg$; h is continuous as a composition of continuous maps:

$$G \xrightarrow{(c_h, id)} G \times G \xrightarrow{\mu} G$$
$$g \mapsto (h, g) \mapsto hg$$

Similarly, the inverse h^{-1} is continuous.

Let H be a closed normal subgroup of G. Then G/H is a group. Let $p: G \to G/H$, $g \mapsto gH$. Then p is a surjection. We equip G/H with the quotient topology from G: U is open in G/H if and only if $p^{-1}(U)$ is open in G. Then p is a continuous function.

Lemma 10.12. The quotient map $p: G \to G/H$ is an open map.

proof Let U be open in G. Let $g \in G$. Then $U_g = \{xg \mid x \in U\}$ is open in G, since $g: G \to G$, $x \mapsto xg$, is a homeomorphism. Now,

$p^{-1}(p(U)) = \bigcup_{h \in H} Uh$ is open in G as a union of open sets. Thus $p(U)$ is open in G/H. $\square$

Proposition 10.13. Let G be a topological group and let H be a closed, normal subgroup of G. Then G/H is a topological group.

proof. Homework $\square$

Proposition 10.14 Let G be a topological group and let H be an open subgroup of G. Then H is closed in G.

proof. Homework $\square$