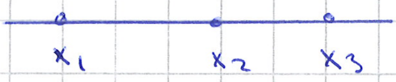
Example Radon's theorem (1905)

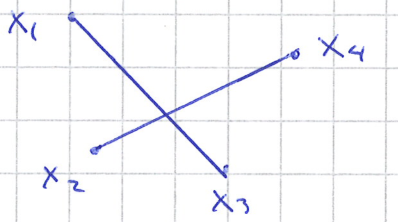
Let $\bar{x}_1, \dots, \bar{x}_{n+2}$ be distinct points in $\mathbb{R}^n$.
Then $\{\bar{x}_1, \dots, \bar{x}_{n+2}\}$ can be partitioned in two subsets S and T such that $[S] \cap [T] \neq \emptyset$.
(Here $[S]$ and $[T]$ denote the convex hulls of S and T, respectively.)

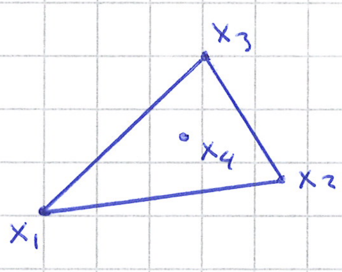
$n=1$



$S = \{x_1, x_3\}$, $T = \{x_2\}$

$n=2$





$S = \{x_1, x_2, x_3\}$
$T = \{x_4\}$

$S = \{x_1, x_3\}$
$T = \{x_2, x_4\}$

proof. Consider the following equations for solving the unknowns $\alpha_1, \dots, \alpha_{n+2}$:

$$\begin{cases} \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \dots + \alpha_{n+2} \bar{x}_{n+2} = 0 \\ \alpha_1 + \alpha_2 + \dots + \alpha_{n+2} = 0 \end{cases}$$

One equation for each coordinate $\Rightarrow n$ equations
+ one more equation $\Rightarrow$ altogether $n+1$ equations

$n+1$ equations, $n+2$ unknowns $\Rightarrow$ the system has a nontrivial solution $(\alpha_1, \dots, \alpha_{n+2})$.

Without loss of generality, we may assume that

$$0 > \alpha_1 \le \alpha_2 \le \dots \le \alpha_k \le 0 \le \alpha_{k+1} \le \dots \le \alpha_{n+2} > 0,$$

for some $k \in \{1, \dots, n+1\}$.

Then $\underbrace{-\alpha_1 - \dots - \alpha_k}_{>0} = \underbrace{\alpha_{k+1} + \dots + \alpha_{n+2}}_{>0}$.

Then

$$\bar{p} = \frac{-\alpha_1}{-(\alpha_1 + \dots + \alpha_k)} \bar{x}_1 + \dots + \frac{-\alpha_k}{-(\alpha_1 + \dots + \alpha_k)} \bar{x}_k$$

$$= \frac{\alpha_{k+1}}{\alpha_{k+1} + \dots + \alpha_{n+2}} \bar{x}_{k+1} + \dots + \frac{\alpha_{n+2}}{\alpha_{k+1} + \dots + \alpha_{n+2}} \bar{x}_{n+2} = \bar{q}$$

Let $S = \{\bar{x}_1, \dots, \bar{x}_k\}$ and $T = \{\bar{x}_{k+1}, \dots, \bar{x}_{n+2}\}$.

Then $\bar{p} \in [S]$ and $q \in [T]$.

$\bar{p} = \bar{q} \Rightarrow [S] \cap [T] \neq \emptyset$. $\square$

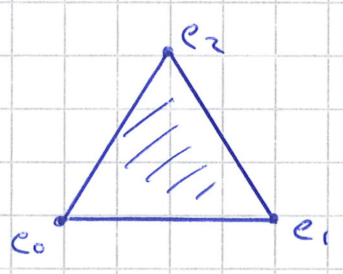
Reformulation of Radon's theorem

For every affine map $\phi : \Delta^{n+1} \to \mathbb{R}^n$, there exist two faces S and T of Δ^{n+1}, $\underbrace{S \cap T}_{=\emptyset}$, such that $\phi(S) \cap \phi(T) \neq \emptyset$.

Here: Δ^{n+1} = the standard $n+1$ simplex

vertices $e_0, \dots, e_{n+1}$

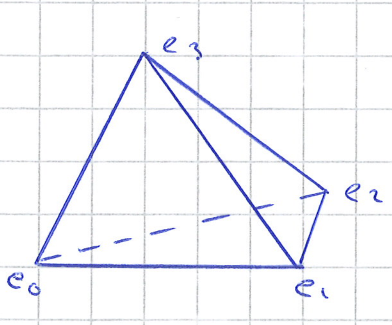
partition of vertices $\Rightarrow$ 2 faces that do not intersect, not necessarily of the same dimension



for example 0-face $\{e_2\}$

1-face $[e_0, e_1]$

$\{e_2\} \cap [e_0, e_1] = \emptyset$



$[e_0, e_1], [e_2, e_3]$ 1-faces

$[e_0, e_1] \cap [e_2, e_3] = \emptyset$

In fact, the following holds although we cannot prove it yet:

<u>Theorem</u> Let $\partial\Delta^{n+1}$ denote the boundary of the standard $n+1$-simplex. For every continuous map $f:\partial\Delta^{n+1}\to\mathbb{R}^n$ there exist two disjoint faces S and T of Δ^{n+1} such that $f(S)\cap f(T)\neq\emptyset$. $\square$

The previous theorem is a version of the following famous result.

<u>Theorem (Borsuk-Ulam)</u> If $f:S^n\to\mathbb{R}^n$ is continuous and $n\geq 1$, then there exists $x\in S^n$ with $f(x)=f(-x)$.

We may later prove the Borsuk-Ulam theorem for $n=2$. The case $n>2$ is harder.

Clearly, we may replace Δ^{n+1} above by any $n+1$-simplex because:

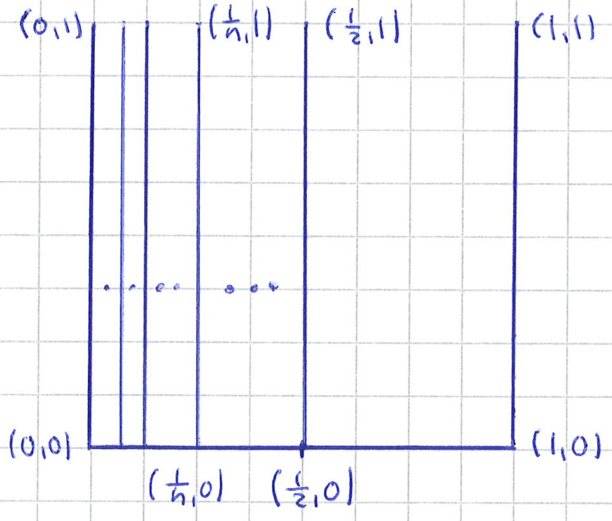
<u>Proposition</u> Any two n-simplexes are homeomorphic via an affine map.

<u>proof.</u> The claim follows immediately from the fact that all affine maps are continuous (HW). $\square$

6. On retracts, deformation retracts etc.

Let $X = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1,\ x = \frac{1}{n} \text{ or } x = 0,\ y = 0,\ 0 \leq x \leq 1\}$

The space X is called the <u>comb space</u>.



The comb space is a good space for examples and counterexamples. For example, it is connected but not locally connected and it is path connected but not locally path connected.

<u>Lemma 6.1</u> The comb space is contractible.

<u>Proof.</u> Define

$$H : X \times I \to X,\quad ((x,y),t) \mapsto \begin{cases} (x, (1-2t)y), & 0 \leq t \leq \frac{1}{2} \\ ((2-2t)x, 0), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

$$t = \tfrac{1}{2}: \left.\begin{array}{l} (x, (1-2\cdot\frac{1}{2})y) = (x,0) \\ ((2-2\cdot\frac{1}{2})x, 0) = (x,0) \end{array}\right\} \text{Gluing lemma} \Rightarrow H \text{ is continuous}$$

$$\left.\begin{array}{l} t = 0: \ H((x,y),0) = (x,y) \\ t = 1: \ H((x,y),1) = (0,0) \end{array}\right\} \Rightarrow \text{the identity map of } X \text{ is null-homotopic, i.e., } X \text{ is contractible.}$$

$\square$

Definition 6.2. Let X be a topological space, let $A \subset X$ and let $i: A \hookrightarrow X$ be the inclusion. If there is a continuous map $r: X \to A$ such that $r \circ i = 1_A$, then A is called a <u>retract</u> of X. In this case the map r is called a <u>retraction</u> of X to A.

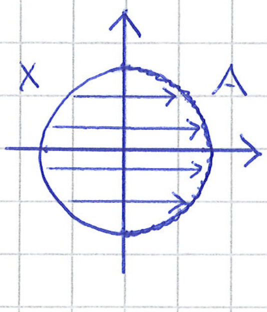
Example 6.3. Let $Y = [0,1] \times [0,1]$ and let X be the comb space. Then $X \subset Y$ and both X and Y are contractible, i.e., they have the same homotopy type. We show that X is not a retract of Y:

Let $r: \underbrace{[0,1]\times[0,1]}_{Y} \to X$ be a continuous map.

Let $(0,y) \in X$, $y \neq 0$. Assume $r(0,y) = (0,y)$.

Let U be a neighborhood of $(0,y)$ in X, we may assume that U is small enough so that it does not intersect the x-axis. Since r is continuous at $(0,y)$, $r(V) \subset U$ for arbitrarily small neighborhoods V of $(0,y)$ in Y. Since Y is locally connected, we may assume that V is connected. Then also $r(V)$ is connected. Therefore, $r(V)$ must lie on the y-axis. Thus $r|X \neq i: X \hookrightarrow Y$ and it follows that r can not be a retraction. $\square$

Example 6.4. Let $X = S^1$ and $A = \{(x,y) \in X \mid x \geq 0\}$. Let $r: X \to A$, $(x,y) \mapsto (|x|, y)$. Then r is a retraction. However, X and A do not have the same homotopy type since A is contractible but X is not (we will prove later that S^1 is not contractible).
Thus A is not a deformation retract of X.



Recall an earlier definition (4.16):

Definition 6.5 (and ~ 4.16) Let X be a topological space and let $A \subset X$. If there is a continuous $F: X \times I \to X$ with

1) $F(x,0) = x$ for all $x \in X$,
2) $F(x,1) \in A$ for all $x \in X$,
3) $F(a,1) = a$ for all $a \in A$,

then A is a *deformation retract* of X. If, in addition, $F(a,t) = a$ for all $a \in A$ and all $t \in I$, then A is called a *strong deformation retract* of X.

Equivalently, A is a deformation retract of X, if there is a continuous $r: X \to A$ with $r \circ i = 1_A$ and $i \circ r \simeq 1_X$ where $i: A \hookrightarrow X$ is the inclusion.

Then:

$$\begin{array}{c} A \text{ is a strong deformation retract of } X \\ ① \Downarrow \qquad \not\Uparrow ④ \\ A \text{ is a deformation retract of } X \\ ② \Downarrow \qquad \not\Uparrow ③ \\ A \text{ is a retract of } X \end{array}$$

① follows from the definitions

② follows from the definitions

③ Example 6.4.

④ Example 6.6.

Be aware: Some authors call a deformation retract what we call a strong deformation retract.

Example 6.6.

Let

$$X = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ x = \tfrac{1}{n},\ \text{or } x=0,\ \text{or } y=0,\ 0 \le x \le 1\}$$

be the comb space. Then the one-point set $\{(0,1)\}$ is a deformation retract of X but not a strong deformation retract of X.

Notice that it does not matter on which interval a homotopy is defined. Then the homotopy

$$F: X \times [0,2] \to X,\ ((x,y),t) \mapsto \begin{cases} (x, (1-2t)y), & 0 \le t \le \frac{1}{2} \\ ((2-2t)x, 0), & \frac{1}{2} \le t \le 1 \\ (0, t-1), & 1 \le t \le 2 \end{cases}$$

deformation retracts X to $\{(0,1)\}$.

We show that $\{(0,1)\}$ is not a strong deformation retract of X. First, recall the tube lemma:

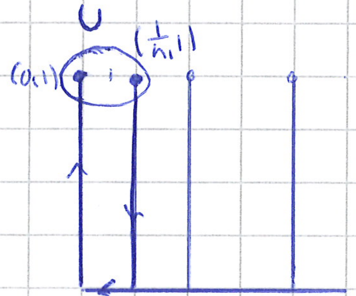
Lemma (Tube lemma) Let X and Y be topological spaces with Y compact. Let N be an open neighborhood in $X \times Y$ of the set $\{x_0\} \times Y$, where $x_0 \in X$. Then there exists an open neighborhood V of x_0 such that $V \times Y \subset N$. (A neighborhood of $\{x_0\} \times Y$ of the form $V \times Y$ is called a tube.) $\square$

Assume $F: X \times I \to X$ is a homotopy that strongly deformation retracts X to the point $(0,1)$. Then

$$F(\{(0,1)\} \times I) = \{(0,1)\},$$
$$F((\tfrac{1}{n},1),1) = (0,1) \quad \forall\, n \in \mathbb{N},$$
$$F((\tfrac{1}{n},1),0) = (\tfrac{1}{n},1) \quad \forall\, n \in \mathbb{N}.$$

Let U be a neighborhood of $(0,1)$. Since F is continuous, the set $\{(0,1)\} \times I$ has a neighborhood N such that $F(N) \subset U$. By the tube lemma, we may assume that N is of the form $N = V \times I$, where V is a neighborhood of $(0,1)$.

Now, $\exists m \in \mathbb{N} : n \geq m \Rightarrow (\frac{1}{n}, 1) \in V$. Then $F((\frac{1}{n},1) \times I) \subset U$. Since $F((\frac{1}{n},1),0) = (\frac{1}{n},1)$, $F((\frac{1}{n},1),1) = (0,1)$ and $F(\{(\frac{1}{n},1)\} \times I)$ is connected, this is impossible if U is an arbitrarily small neighborhood of $(0,1)$: a path from $(\frac{1}{n},1)$ to $(0,1)$ must go through the x-axis and can not stay in an arbitrarily small neighborhood U. $\square$



7. The fundamental groupoid

Definition 7.1: Let X be a topological space and let $f, g: I \to X$ be paths with $f(1) = g(0)$. Define a path $f * g: I \to X$ by

$$(f*g)(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Gluing lemma $\Rightarrow$ $f*g$ is continuous.
Let $[f]$ denote the homotopy class of the path f.
Define an operation among the homotopy classes by setting $[f][g] = [f*g]$.

Lemma 7.2: Let X and Y be topological spaces. Assume X is contractible and Y is path connected. Then any two continuous maps $f, g: X \to Y$ are homotopic (and each is nullhomotopic).

proof. Since X is contractible, there exist $x_0 \in X$ and a homotopy $F: 1_X \simeq C_{x_0}$, where C_{x_0} denotes the constant map at x_0. Then $f \circ F: f \simeq C_{f(x_0)}$ and $g \circ F: g \simeq C_{g(x_0)}$. Since Y is pathconnected, there is a path $h: I \to Y$, $h(0) = f(x_0)$ and $h(1) = g(x_0)$.
Let $H: Y \times I \to Y$, $(y, t) \mapsto h(t)$. Then H is continuous.
The map

$$H \circ (f \times 1_I): X \times I \longrightarrow Y \times I \longrightarrow Y$$
$$(x,t) \longmapsto (f(x), t) \longmapsto h(t)$$

or choose
$$H: X \times I \xrightarrow{pr} I \xrightarrow{h} Y$$
$$(x,t) \mapsto t \mapsto h(t)$$

is a homotopy from $C_{f(x_0)}$ to $C_{g(x_0)}$. Then

$$f \simeq C_{f(x_0)} \simeq C_{g(x_0)} \simeq g. \quad \square$$

Assume X is path-connected. Since I is contractible, Lemma 7.2 implies that all paths $I \to X$ are homotopic. Therefore there is only one homotopy class of maps $I \to X$.

<u>Definition 7.3.</u> Let $A \subset X$ and let $f_0, f_1 : X \to Y$ be continuous maps. Assume $f_0|A = f_1|A$. If there is a continuous map

$$F : X \times I \to Y \quad \text{with} \quad F : f_0 \simeq f_1$$

and

$$F(a,t) = f_0(a) = f_1(a) \quad \forall\, a \in A, \forall\, t \in I,$$

we write $\quad F : f_0 \simeq f_1 \text{ rel } A.$

The homotopy F is called a <u>homotopy rel A</u> or a <u>relative homotopy.</u>

For $A = \emptyset$ we get our earlier definition of a homotopy, which is also called a <u>free homotopy.</u>

<u>Exercise 7.4.</u> Let $A \subset X$. Show that homotopy rel A is an equivalence relation on the set of continuous maps $X \to Y$.

<u>Definition 7.5.</u> Let $\dot{I} = \{0,1\}$. The equivalence class of a path $f : I \to X$ rel $\dot{I}$ is called the <u>path class</u> of f and denoted by $[f]$.

<u>Theorem 7.6.</u> Let f_0, f_1, g_0, g_1 be paths in X. Assume

$$f_0 \simeq f_1 \text{ rel } \dot{I} \quad \text{and} \quad g_0 \simeq g_1 \text{ rel } \dot{I}.$$

If $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 * g_0 \simeq f_1 * g_1 \text{ rel } \dot{I}$

proof. Let $F : f_0 \simeq f_1$ rel $\dot{I}$ and $G : g_0 \simeq g_1$ rel $\dot{I}$.
Let

$$H : I \times I \to X \;,\; (t,s) \mapsto \begin{cases} F(2t,s) \,, & 0 \le t \le \frac{1}{2} \\ G(2t-1,s) \,, & \frac{1}{2} \le t \le 1 \end{cases}$$

Gleuing lemma $\Rightarrow$ H is continuous $\quad (F(1,s) = G(0,s) \; \forall s \in I)$
Then $H : f_0 * g_0 \simeq f_1 * g_1$ rel $\dot{I}$. □

Definition 7.7. Let $x_0, x_1 \in X$ and let $f : I \to X$ be a path, $f(0) = x_0$, $f(1) = x_1$.
Then x_0 is called the origin of f $(x_0 = \alpha(f))$ and x_1 is called the end of f $(x_1 = \omega(f))$.
A path f is closed at x_0 if $\alpha(f) = x_0 = \omega(f)$.

Notice: Let $f, g : I \to X$ be paths, $f \simeq g$ rel $\dot{I}$. Then $\alpha(f) = \alpha(g)$ and $\omega(f) = \omega(g)$. Then it makes sense to speak about the origin and end of a path class, those are denoted by $\alpha[f]$ and $\omega[f]$, respectively.

Definition 7.8. Let $p \in X$. The constant function $i_p : I \to X$, $t \mapsto p$. $\forall t \in I$, is called the constant path at p. The inverse path of a path $f : I \to X$ is the path $f^{-1} : I \to X$, $t \mapsto f(1-t)$.

Theorem 7.9. Let X be a topological space. The set of all path classes in X forms an algebraic system called a groupoid under the operation $[f][g] = [f * g]$ (not always defined) satisfying the following properties:

1) Each path class $[f]$ has an origin $\alpha[f] = p \in X$ and an end $\omega[f] = q \in X$ and

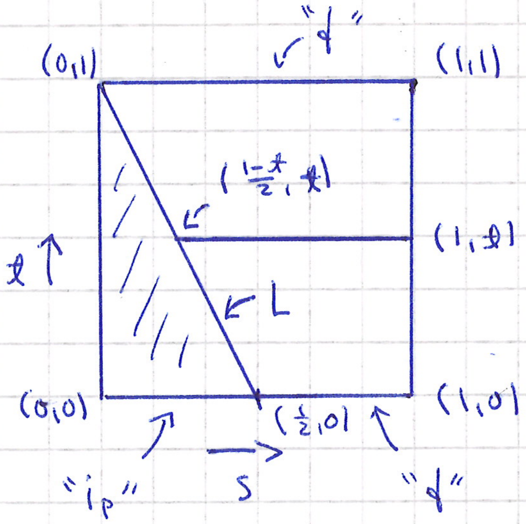
$$[i_p][f] = [f] = [f][i_q].$$

2) Associativity holds whenever possible.

3) If $p = \alpha[f]$ and $q = \omega[f]$, then

$$[f][f^{-1}] = [i_p] \text{ and } [f^{-1}][f] = [i_q].$$

<u>proof.</u> 1) We show that $i_p * f \simeq f$ rel $\dot{I}$. Similarly one can show that $f * i_q \simeq f$ rel $\dot{I}$.
The following picture explains the construction of a suitable homotopy:



Equation for L: $t = 1-2s$
$\Rightarrow s = \frac{1-t}{2}$

For every $t \in [0,1)$, stretch the interval $[\frac{1-t}{2}, 1]$ to $[0,1]$.

Exercise: Stretching can be done by an affine map θ_t:

$$\theta_t : [\tfrac{1-t}{2}, 1] \to [0,1], \quad \theta_t(s) = \frac{s-(1-t)/2}{1-(1-t)/2}.$$

(Then: $\theta_t(\frac{1-t}{2}) = 0$, $\theta_t(1) = 1$ o.k.)

Let $H : I \times I \to X$,

$$H(s,t) = \begin{cases} p, & \text{if } 2s \leq 1-t \quad (\text{i.e. in } \triangle) \\ f(\theta_t(s)) = f\left(\frac{2s-1+t}{1+t}\right), & \text{if } 2s \geq 1-t. \end{cases}$$

$2s = 1-t$: $f(\theta_t(s)) = f(0) = p$, Gluing lemma $\Rightarrow$ H is continuous.

$$t=0: \; H(s,0) = \begin{cases} p, & \text{if } 2s \leq 1 \\ f(2s-1), & \text{if } 2s \geq 1 \end{cases} \quad \Rightarrow H(s,0) = i_p * f$$

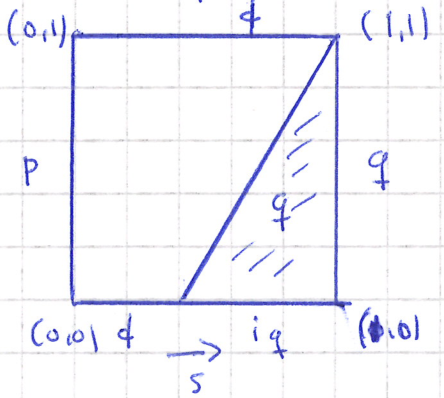
$t=1$: $H(s,1) = f\left(\frac{2s}{2}\right) = f(s) \Rightarrow H(\ ,1) = f$

$s=0$: $H(0,t) = p \quad \forall t$

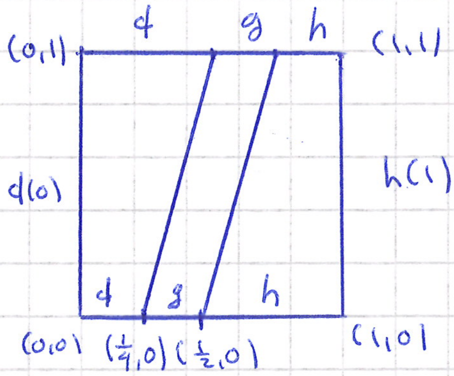
$s=1$: $H(1,t) = f\left(\frac{2-1+t}{1+t}\right) = f(1) = q \quad \forall t$

Thus $H : i_p * f \simeq f \text{ rel } \dot{I}$

$f * i_q \simeq f \text{ rel } \dot{I}$



2) Let $f, g, h : I \to X$.
Assume $f(1) = g(0)$ and $g(1) = h(0)$.
Then $f*(g*h)$ and $(f*g)*h$ are defined.



$$(f*g)*h = \begin{cases} f(4s), & 0 \le s \le \frac{1}{4} \\ g(4s-1), & \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

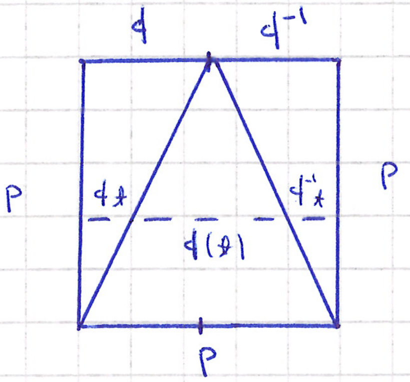
$$f*(g*h) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(4s-2), & \frac{1}{2} \le s \le \frac{3}{4} \\ h(4s-3), & \frac{3}{4} \le s \le 1 \end{cases}$$

As in part 1, one can write a homotopy
$H : (f*g)*h \simeq f*(g*h) \text{ rel } \dot{I}$.

3) We show that $i_p \simeq f * f^{-1}$ rel $\dot{I}$.

$$f * f^{-1}(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ f^{-1}(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ f(2-2s), & \frac{1}{2} \le s \le 1 \end{cases} \qquad (\text{since } f^{-1}(t) = f(1-t))$$



Let $f_t = f|[0,t]$
and $f^{-1}_t = f^{-1}|[1-t,1]$

Let $H(s,t) = \begin{cases} f(2s), & 0 \le s \le \frac{t}{2} \\ f(t), & \frac{t}{2} \le s \le 1 - \frac{t}{2} \\ f(2-2s), & 1 - \frac{t}{2} \le s \le 1 \end{cases}$

$s = t/2$: $f(2 \cdot \frac{t}{2}) = f(t)$

$s = 1 - \frac{t}{2}$: $f(2 - 2(1 - t/2)) = f(t)$

$\Bigg\}$ Gluing lemma $\Rightarrow$ H is continuous

$H(s,0) = f(0) = p$

$H(s,1) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ f(2-2s), & \frac{1}{2} \le s \le 1 \end{cases}$
$= f * f^{-1}(s)$

$H(0,t) = f(0) = p \quad \forall t$

$H(1,t) = f(2-2\cdot 1) = f(0) = p \quad \forall t$

Thus $H : i_p \simeq f * f^{-1}$ rel $\dot{I}$

$\square$

Definition 7.10. Choose $x_0 \in X$ and call it the *basepoint*. The *fundamental group* of X with basepoint x_0 is

$$\pi_1(X, x_0) = \{[f] : [f] \text{ is a path class in } X \text{ with } \alpha[f] = x_0 = \omega[f]\}$$

with the operation $[f][g] = [f * g]$.

Theorem 7.11. $\pi_1(X, x_0)$ is a group for each $x_0 \in X$.

$\square$