5. Simplexes and affine spaces

Definition 5.1. Let $A \subset \mathbb{R}^n$. If for all $x, x' \in A$, $x \neq x'$, the line determined by x and x' is contained in A, we call A an <u>affine</u> set.

Notice: 1) $\emptyset$ and one-point subsets are affine

2) All affine sets are convex.

Theorem 5.2. Let X_j, $j \in J$, be affine (or convex) subsets of $\mathbb{R}^n$. Then also $\bigcap_{j \in J} X_j$ is affine (or convex). $\square$

Let $X \subset \mathbb{R}^n$. The <u>affine</u> (or <u>convex</u>) <u>hull</u> of X is the intersection of all affine (or convex) subsets of $\mathbb{R}^n$ containing X. We also say that the affine (or convex) hull of X is the <u>affine</u> (or <u>convex</u>) <u>set spanned by X</u>.

Notation: $[X]$ = the convex hull of X

Definition 5.3. Let $p_0, p_1, \ldots, p_m \in \mathbb{R}^n$. An <u>affine combination</u> of $p_0, p_1, \ldots, p_m$ is a point x with

$$x = t_0 p_0 + t_1 p_1 + \ldots + t_m p_m,$$

where $\sum_{i=0}^{m} t_i = 1$. A <u>convex combination</u> is an affine combination such that $t_i \geq 0 \ \forall i$.

Theorem 5.4. Let $p_0, p_1, \ldots, p_m \in \mathbb{R}^n$. The convex hull $[p_0, p_1, \ldots, p_m]$ of the set $\{p_0, p_1, \ldots, p_m\}$ is the set of all convex combinations of $p_0, p_1, \ldots, p_m$.

proof. Let S be the set of all convex combinations of $p_0, p_1, \ldots, p_m$.

1) $[p_0, \ldots, p_m] \subset S$: Suffices to show: S is convex and $p_0, p_1, \ldots, p_m \in S$.

Let $j \in \{0, \ldots, m\}$, set $t_j = 1$, $t_i = 0$ for $i \neq j$. Then $p_j = \sum_{i=0}^{m} t_i p_i \in S$. $\therefore$ $p_0, p_1, \ldots, p_m \in S$

Let then $\alpha = \sum a_i p_i$ and $\beta = \sum b_i p_i$, where $a_i, b_i \geq 0$, $\sum a_i = 1$, $\sum b_i = 1$. Let $t \in I$. Then

$$t\alpha + (1-t)\beta = t \sum_{i=0}^{m} a_i p_i + (1-t) \sum_{i=0}^{m} b_i p_i$$

$$= \sum_{i=0}^{m} [t a_i + (1-t) b_i] p_i \in S,$$

since $t a_i + (1-t) b_i \geq 0 \ \forall i$ and

$$\sum_{i=0}^{m} [t a_i + (1-t) b_i] = t \sum_{i=0}^{m} a_i + (1-t) \sum_{i=0}^{m} b_i = t + (1-t) = 1.$$

$\therefore$ S is convex

2) $S \subset [p_0, \ldots, p_m]$: We show that $S \subset X$ for any convex subset X of $\mathbb{R}^n$ containing $\{p_0, p_1, \ldots, p_m\}$. The proof is done by induction on m:

i) Let $m = 0$. Then $S = \{p_0\}$ and we are done.

ii) Let $m > 0$. Let $t_i \geq 0$ and $\sum t_i = 1$. We may assume that $t_0 \neq 1$ (if $t_0 = 1$, then $\sum_{i=0}^{m} t_i p_i = p_0 \in X$). Let $p = \sum_{i=0}^{m} t_i p_i$. Then

$$q = \underbrace{\frac{t_1}{1-t_0} p_1 + \ldots + \frac{t_m}{1-t_0} p_m}_{\Uparrow} \in X \quad \text{by induction.}$$

convex combination: a) $\frac{t_i}{1-t_0} \geq 0 \ \forall i$

b) $$\sum_{i=1}^{m} \frac{t_i}{1-t_0} = \frac{\sum_{i=1}^{m} t_i}{1-t_0} = \frac{1-t_0}{1-t_0} = 1.$$

Then $p = t_0 p_0 + (1-t_0) q \in X$, since X is convex.

$\square$

<u>Corollary 5.5.</u> The affine set spanned by $\{p_0, p_1, \dots, p_m\}$ is the set of all affine combinations of $p_0, p_1, \dots, p_m$.

<u>proof.</u> Similar to the proof of Theorem 5.4. $\square$

<u>Definition 5.6.</u> An ordered set $\{p_0, p_1, \dots, p_m\} \subset \mathbb{R}^n$ is called <u>affine independent</u>, if $\{p_1 - p_0, \dots, p_m - p_0\}$ is a linearly independent subset of $\mathbb{R}^n$.

<u>Notice:</u> 1) $\{p_0\}$ is affine independent
2) $\emptyset$ is linearly independent
3) $\{p_0, p_1\}$ is affine independent if $p_1 - p_0 \neq 0$, i.e. if, $p_1 \neq p_0$
4) linearly indep. set is affine independent
5) Let $\{p_1, \dots, p_m\}$ be lin. indep. Then $\{0, p_1, \dots, p_m\}$ is affine indep. but not lin. indep.

<u>Theorem 5.7.</u> Let $\{p_0, p_1, \dots, p_m\}$ be an ordered set of points in $\mathbb{R}^n$. The following conditions are equivalent:

1) $\{p_0, p_1, \dots, p_m\}$ is affine independent

2) If $\{s_0, s_1, \dots, s_m\} \subset \mathbb{R}$ satisfies $\sum_{i=0}^{m} s_i p_i = 0$ and $\sum_{i=0}^{m} s_i = 0$, then $s_0 = s_1 = \dots = s_m = 0$.

3) Every element x of the affine set spanned by $\{p_0, p_1, \dots, p_m\}$ has a unique expression as an affine combination:

$$x = \sum_{i=0}^{m} t_i p_i \quad \text{where} \quad \sum_{i=0}^{m} t_i = 1.$$

proof. We show that $1 \Rightarrow 2$, $2 \Rightarrow 3$ and $3 \Rightarrow 1$.

$1 \Rightarrow 2$: Assume $\sum s_i = 0$ and $\sum s_i p_i = 0$. Then

$$0 = \sum_{i=0}^{m} s_i p_i = \sum_{i=0}^{m} s_i p_i - \underbrace{\left(\sum_{i=0}^{m} s_i\right)}_{=0} p_0$$

$$= \sum_{i=0}^{m} s_i (p_i - p_0) = \sum_{i=1}^{m} s_i (p_i - p_0), \text{ since } p_0 - p_0 = 0.$$

$\{p_0, \dots, p_m\}$ is affine indep. $\Rightarrow \{p_1 - p_0, \dots, p_m - p_0\}$ is lin. indep. $\Rightarrow s_i = 0 \ \forall i \in \{1, \dots, m\}$.
Since $\sum_{i=0}^{m} s_i = 0$, must be $s_0 = 0$ as well.

$2 \Rightarrow 3$: Let A denote the affine set spanned by $\{p_0, p_1, \dots, p_m\}$. Let $x \in A$. By Corollary 5.5,

$$x = \sum_{i=0}^{m} t_i p_i, \quad \text{where } \sum_{i=0}^{m} t_i = 1.$$

Assume that also $x = \sum_{i=0}^{m} t_i' p_i$, where $\sum_{i=0}^{m} t_i' = 1$.
Then

$$0 = \sum_{i=0}^{m} (t_i - t_i') p_i .$$

Now, $\sum (t_i - t_i') = \sum t_i - \sum t_i' = 1 - 1 = 0$. By Condition 2, $t_i - t_i' = 0$ for all i, i.e., $t_i = t_i'$ for all i.

$3 \Rightarrow 1$: If $m = 0$, there is nothing to prove. Therefore, assume $m > 0$. Assume that each $x \in A$ has a unique expression as an affine combination of $p_0, p_1, \dots, p_m$. Let's make a counter assumption, by assuming that $\{p_0, p_1, \dots, p_m\}$ is not affine independent. Then $\{p_1 - p_0, \dots, p_m - p_0\}$ is linearly dependent. It follows that there are $r_i \in \mathbb{R}$, not all of them 0, such that

$$0 = \sum_{i=1}^{m} r_i (p_i - p_0). \qquad (*)$$

Assume $r_j \neq 0$. By multiplying $(*)$ by $\frac{1}{r_j}$ if necessary, we may assume that $r_j = 1$.

There now are two different ways to write p_j as an affine combination of $p_0, p_1, \dots, p_m$:

$$p_j = 1 \cdot p_j$$

$$p_j = -\sum_{i \neq j} r_i (p_i - p_0) + p_0$$

$$= -\sum_{i \neq j} r_i p_i + \Big(1 + \sum_{i \neq j} r_i\Big) p_0 .$$

A contradiction. Thus $\{p_0, p_1, \dots, p_m\}$ is affine indep. $\square$

<u>Corollary 5.8.</u> Affine independence is a property of the set $\{p_0, p_1, \dots, p_m\}$ that is independent of the given ordering. $\square$

<u>Corollary 5.9.</u> Let A be the affine set in $\mathbb{R}^n$ spanned by an affine independent set $\{p_0, p_1, \dots, p_m\}$. Then A is of the form

$$A = x_0 + V,$$

where $x_0 \in \mathbb{R}^n$ and V is an m-dimensional vector subspace of $\mathbb{R}^n$.

<u>proof.</u> Let V be the vector subspace of $\mathbb{R}^n$ whose basis is $\{p_1 - p_0, \dots, p_m - p_0\}$. Choose $x_0 = p_0$. $\square$

<u>Definition 5.10.</u> A set $\{a_1, a_2, \dots, a_k\}$ of points in $\mathbb{R}^n$ is in <u>general position</u>, if every subset of it consisting on $n+1$ points is affine independent.

<u>Notice:</u> Assume $\{a_1, \dots, a_k\} \subset \mathbb{R}^n$ is in general position.

$n=1$: every pair $\{a_i, a_j\}$ is affine indep., i.e., $a_i \neq a_j$ for $i \neq j$.

$n=2$: Three-point sets $\{a_i, a_j, a_k\}$ are affine indep. $\Leftrightarrow$ $\{a_j - a_i, a_k - a_i\}$ are lin. indep. This means that no three points a_i, a_j, a_k can lie in a single straight line.

$n=3$: No four points of $\{a_0, \dots, a_m\}$ can lie in a single plane.

Definition 5.11. Let $\{p_0, p_1, \dots, p_m\}$ be an affine indep. subset of $\mathbb{R}^n$. Let A be the affine set spanned by $\{p_0, p_1, \dots, p_m\}$. Let $x \in A$ and let $(t_0, t_1, \dots, t_m)$ be the unique $(m+1)$-tuple with $\sum_{i=0}^{m} t_i = 1$ and $x = \sum_{i=0}^{m} t_i p_i$ (such a tuple exists by Theorem 5.7). The numbers $t_0, t_1, \dots, t_m$ are called the barycentric coordinates of x (relative to the ordered set $\{p_0, p_1, \dots, p_m\}$).

Definition 5.12. Let $\{p_0, p_1, \dots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$. The convex set $[p_0, p_1, \dots, p_m]$ spanned by $\{p_0, p_1, \dots, p_m\}$ is called the (affine) m-simplex with vertices $p_0, p_1, \dots, p_m$.

Theorem 5.13. Let $\{p_0, p_1, \dots, p_m\}$ be an affine independent set. Then every $x \in [p_0, p_1, \dots, p_m]$ has a unique expression of the form

$$x = \sum t_i p_i, \quad \text{where} \quad \sum t_i = 1 \text{ and } t_i \geq 0 \ \forall i.$$

proof. By Theorem 5.4, every $x \in [p_0, p_1, \dots, p_m]$ has an expression of such form. By Theorem 5.7, the expression is unique. $\square$

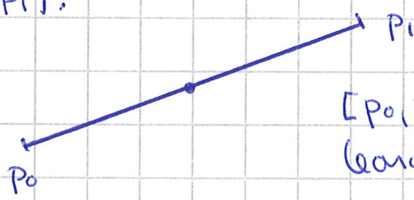
Definition 5.14. Let $\{p_0, p_1, \dots, p_m\}$ be an affine independent set. The barycenter of $[p_0, p_1, \dots, p_m]$ is

$$\frac{1}{m+1}(p_0 + p_1 + \dots + p_m).$$

Example

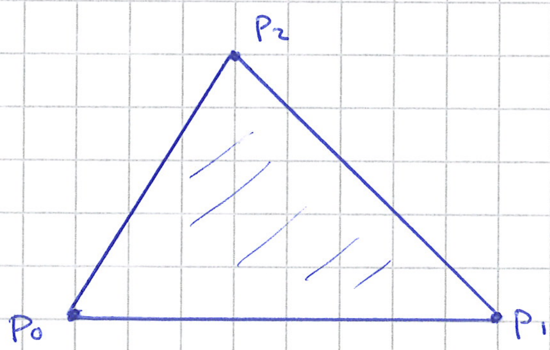
1) $[p_0]$ is a 0-simplex, consists of one point, which is its own barycenter.

2) The one-simplex $[p_0, p_1] = \{ t p_0 + (1-t) p_1 \mid t \in I \}$:
Barycenter $= \frac{1}{2}(p_0 + p_1)$.



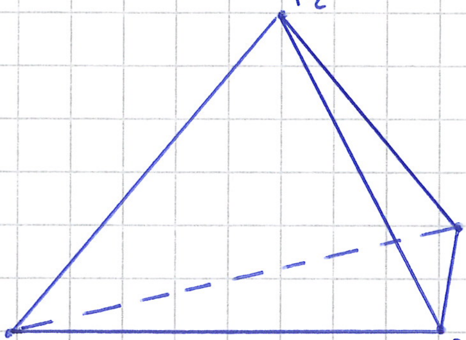
$[p_0, p_1]$ = line segment
barycenter = midpoint of the line segment

3) The 2-simplex $[p_0, p_1, p_2]$:
A triangle (with interior), vertices p_0, p_1, p_2.
Barycenter $= \frac{1}{3}(p_0 + p_1 + p_2)$ = the center of gravity of $[p_0, p_1, p_2]$
Edges: $[p_0, p_1]$, $[p_0, p_2]$, $[p_1, p_2]$



Consider the edge $[p_0, p_1]$.
The barycentric coordinates of $x \in [p_0, p_1]$ are of the form $(t, 1-t, 0)$.
Generally, $x \in [p_0, p_1, p_2]$ lies on an edge if and only if one of it's barycentric coordinates is 0.

4) The 3-simplex $[p_0, p_1, p_2, p_3]$:
A solid tetrahedron, vertices p_0, p_1, p_2, p_3.
The triangular face opposite of p_i consists of all those points on $[p_0, p_1, p_2, p_3]$ whose ith barycentric coordinate is 0.



Example (Standard n-simplex)

Standard basis vectors for $\mathbb{R}^{n+1}$: $e_0, \dots, e_n$

e_i: the $(i+1)$st (cartesian) coordinate equals 1, all other coordinates are 0.

The set $\{e_0, e_1, \dots, e_n\}$ is linearly independent, hence also affine independent. The n-simplex $[e_0, e_1, \dots, e_n]$ is called the <u>standard n-simplex</u> and denoted by Δ^n. It consists of all convex combinations $x = \sum t_i e_i$. In this case, barycentric and cartesian coordinates coincide.

<u>Definition 5.15.</u> Let $[p_0, p_1, \dots, p_m]$ be an m-simplex. The <u>face opposite p_i</u> is the $(m-1)$-simplex $[p_0, \dots, \widehat{p_i}, \dots, p_m] = \{ \sum t_j p_j \mid t_j \geq 0, \sum t_j = 1, t_i = 0 \}$. The <u>boundary</u> of $[p_0, p_1, \dots, p_m]$ is the union of its faces.

Notice: The notation $[p_0, \ldots, \hat{p_i}, \ldots, p_m]$ means that the vertex $\hat{p_i}$ is deleted.

An m-simplex has $m+1$ faces.

Let $0 \le k \le m-1$. A k-simplex spanned by $k+1$ of the vertices $\{p_0, p_1, \ldots, p_m\}$ is called a <u>k-face</u> of $[p_0, p_1, \ldots, p_m]$. Then the faces defined as in Def. 5.15 can be called $(m-1)$-faces.

The following theorem will be needed later:

Theorem 5.16. Let S be the n-simplex $[p_0, p_1, \ldots, p_n]$. Then:

1) If $u, v \in S$, then $\|u-v\| \le \sup_i \|u-p_i\|$

2) $\operatorname{diam} S = \sup_{i,j} \|p_i - p_j\|$

3) If b is the barycenter of S, then

$$\|b - p_i\| \le \frac{n}{n+1} \operatorname{diam} S.$$

Here $\operatorname{diam} S$ denotes the diameter of S, which by definition is the set $\sup\{\|u-v\| \mid u, v \in S\}$.

proof.

1) Write $v = \sum t_i p_i$, where $t_i \ge 0$ and $\sum t_i = 1$. Then

$$\|u-v\| = \|u - \sum t_i p_i\| = \|(\sum t_i) u - \sum t_i p_i\|$$

$$= \|\sum t_i (u-p_i)\| \le \sum t_i \|u-p_i\|$$

$$\le \sum t_i \sup \|u-p_i\| = \sup_i \|u-p_i\|.$$

2) $\|u-v\| \leq \sup_i \|u-p_i\| \leq \sup_i (\sup_j \|p_j-p_i\|)$

$$= \sup_{i,j} \|p_j - p_i\|.$$

3) $$b = \frac{1}{n+1} \sum p_i,$$

$$\|b - p_i\| = \| \sum_{j=0}^{n} \frac{1}{n+1} p_j - p_i \| = \| \sum_{j=0}^{n} \frac{1}{n+1} p_j - \underbrace{\left(\sum_{j=0}^{n} \frac{1}{n+1} \right)}_{=1} p_i \|$$

$$= \| \sum_{j=0}^{n} \frac{1}{n+1} (p_j - p_i) \| \leq \sum_{j=0}^{n} \frac{1}{n+1} \|p_j - p_i\|$$

$$= \frac{1}{n+1} \sum_{j=0}^{n} \|p_j - p_i\|$$

$$\leq \frac{n}{n+1} \sup_{i,j} \|p_j - p_i\| \quad (\text{since } \|p_j - p_i\| = 0 \text{ if } i = j)$$

$$= \frac{n}{n+1} \operatorname{diam} S. \quad \square$$

Affine maps

Definition 5.17. Let A be the set spanned by the affine independent set $\{p_0, p_1, \dots, p_m\}$. Let $k \geq 1$. An affine map $T: A \to \mathbb{R}^k$ is a function satisfying

$$T(\sum t_j p_j) = \sum t_j T(p_j)$$

whenever $\sum t_j = 1$.

Affine maps preserve affine combinations, hence also convex combinations. Also the restriction of T to the convex hull $[p_0, p_1, \dots, p_m]$ is called an affine map. An affine map is determined by its values on an affine independent subset. Then its restriction to a simplex is determined by its values on the vertices.

Theorem 5.18. Let $[p_0, \dots, p_m]$ be an m-simplex and let $[q_0, \dots, q_n]$ be an n-simplex. Let $f : \{p_0, \dots, p_m\} \to [q_0, \dots, q_n]$ be any function. Then there is a unique affine map $T : [p_0, \dots, p_m] \to [q_0, \dots, q_n]$ satisfying $T(p_i) = f(p_i)$ for $i = 0, 1, \dots, m$.

Proof. Define $T(\sum t_i p_i) = \sum t_i f(p_i)$ for convex combinations $\sum t_i p_i$. Uniqueness is clear. $\square$