2) $\Rightarrow$ 3): Assume that $g: D^{n+1} \to Y$ extends f.
Let

$$F: S^n \times I \to Y, \ (x,t) \mapsto g(\underbrace{(1-t)x + t x_0}_{\in D^{n+1}}).$$

Clearly, F is continuous. For all $x \in S^n$,

$$F(x,0) = g(x) = f(x) \text{ and } F(x,1) = g(x_0) = f(x_0).$$

Thus $F: f \simeq k$, where $k: S^n \to Y$, $x \mapsto f(x_0)$.
For all $t \in I$,

$$F(x_0,t) = g((1-t)x_0 + t x_0) = g(x_0) = f(x_0).$$

3) $\Rightarrow$ 1) obvious $\square$

3. Convexity, contractibility and cones

__Definition 3.1.__ A subset X of $\mathbb{R}^m$ is called __convex__, if $tx + (1-t)y \in X$ for all $x,y \in X$ and for all $t \in I$.

__Definition 3.2.__ A topological space X is called __contractible__ if the identity map $1_X : X \to X$ is nullhomotopic.

__Theorem 3.3.__ Every convex set is contractible.

__proof.__ Let X be a convex set and let $x_0 \in X$. Define $c: X \to X$ by $c(x) = x_0$ for all $x \in X$. Define $F: X \times I \to X$ by $F(x,t) = t x_0 + (1-t)x$. Then $F: 1_X \simeq c$. $\square$

<u>Definition 3.4.</u> Let X be a topological space and let $X' = \{X_j \mid j \in J\}$ be a collection of subsets of X. We call X' a <u>partition</u> of X, if the following hold:

1) $X_j \neq \emptyset$, for every $j \in J$

2) $X = \bigcup_{j \in J} X_j$

3) $X_i \cap X_j = \emptyset$, for all $i, j \in J$, $i \neq j$

<u>Definition 3.5.</u> Let X be a topological space and let $X' = \{X_j \mid j \in J\}$ be a partition of X. The map

$$\nu : X \to X', \quad x \mapsto X_j \quad \text{if } x \in X_j,$$

is called the <u>natural map</u> (or <u>natural projection</u> or <u>quotient map</u>). The <u>quotient topology</u> on X' is the collection of all subsets U' of X' such that $\nu^{-1}(U')$ is open in X.

Notice that the natural map ν is always a continuous surjection.

<u>Example</u> Let X, Y be topological spaces and let $f : X \to Y$ be a function. Define $x \sim x'$ if $f(x) = f(x')$. Then $\sim$ is an equivalence relation. Let $\nu : X \to X/\ker f$ be the natural map, where $X/\ker f$ denotes the quotient space. Let $[x]$ denote the equivalence class of $x \in X$. The map

$$\varphi : X/\ker f \to Y, \quad [x] \mapsto f(x),$$

is an injection making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\nu"'] & Y \\ X/\ker f \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commute.

<u>Definition 3.6.</u> A continuous surjection $f: X \to Y$ is called an identification if it satisfies the following: A subset U of Y is open if and only if $f^{-1}(U)$ is open in X.

<u>Example</u> Let $\sim$ be an equivalence relation on a topological space X and let $X/\sim$ be equipped with the quotient topology. Then the natural map $\nu: X \to X/\sim$ is an identification.

<u>Example</u> Let $f: X \to Y$ be a continuous map that is a surjection. If f is either open or closed, then it is an identification.

Let $f: X \to Y$ be a continuous map. Assume there is a continuous map $s: Y \to X$ with $f \circ s = 1_Y$. We call s a <u>section</u> of f. Notice that f must be surjective in order to have a section.

<u>Example</u> A continuous map $f: X \to Y$ having a section is an identification.

Theorem 3.7. Let $f: X \to Y$ be a continuous surjection. Then f is an identification if and only if the following holds: For all spaces Z and for all functions $g: Y \to Z$, the function g is continuous if and only if $g \circ f$ is continuous.

proof.

$$\begin{tikzcd} X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

Assume first that f is an identification. Then $g \circ f$ is continuous, if g is continuous. Assume $g \circ f$ is continuous. Let V be an open set in Z. Then

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$$

is open in X. Since f is an identification, it follows that $g^{-1}(V)$ is open in Y. Since V was chosen arbitrarily, it follows that g is continuous.

Assume then that the condition holds. Let $Z = X/\ker f$ and let $\nu: X \to X/\ker f$ be the natural map. Let $\varphi: X/\ker f \to Y$ be the injection $[x] \mapsto f(x)$. Since f is a surjection, also φ is a surjection. The diagram

$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[dr, "f"'] & X/\ker f \arrow[d, shift left, "\varphi"] \\ & Y \arrow[u, shift left, "\varphi^{-1}"] \end{tikzcd}$$

commutes. Now, $\nu = \varphi^{-1} \circ f$ is continuous. Since the condition holds, it follows that φ^{-1} is continuous. Since ν is an identification, it follows that φ is continuous. Then φ is a homeomorphism and, consequently, f is an identification. □

<u>Definition 3.8.</u> Let $f: X \to Y$ be a function and let $y \in Y$. Then $f^{-1}(y)$ is called the <u>fiber</u> over y.

<u>Corollary 3.9.</u> Let X, Y and Z be topological spaces and let $f: X \to Y$ be an identification. Let $h: X \to Z$ be a continuous function that is constant on each fiber of f. Then $g: Y \to Z$, $y \mapsto h(f^{-1}(y))$, is continuous. Moreover, the following are equivalent:

1) g is an open (closed) map

2) $h(U)$ is open (closed) in Z whenever U is an open (closed) set in X with $U = f^{-1}(f(U))$.

<u>proof</u>

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

The function g is well defined, since h is constant on each fiber of f. The function $g \circ f = h$ is continuous. Therefore, by Theorem 3.7, also g is continuous.

Assume then that g is an open map. Let U be an open subset of X with $U = f^{-1}(f(U))$. Since f is an identification, it follows that

$f(U)$ is open in Y. Since g is open, it follows that $g(f(U))$ is open in Z. But then, $h(U) = g(f(U))$ is open in Z. Thus $1 \Rightarrow 2$.

Finally, assume that condition 2 holds. Let V be an open subset of Y. Since f is a surjection, $f(f^{-1}(V)) = V$. Thus $f^{-1}(f(f^{-1}(V))) = f^{-1}(V)$. By condition 2, $g(V) = h(f^{-1}(V))$ is open. Thus g is an open map and $2 \Rightarrow 1$. $\square$

<u>Corollary 3.10.</u> Let X and Z be topological spaces and let $h: X \to Z$ be an identification. Then the map

$$\varphi : X/\ker h \longrightarrow Z, \quad [x] \longmapsto h(x),$$

is a homeomorphism.

<u>Proof.</u>

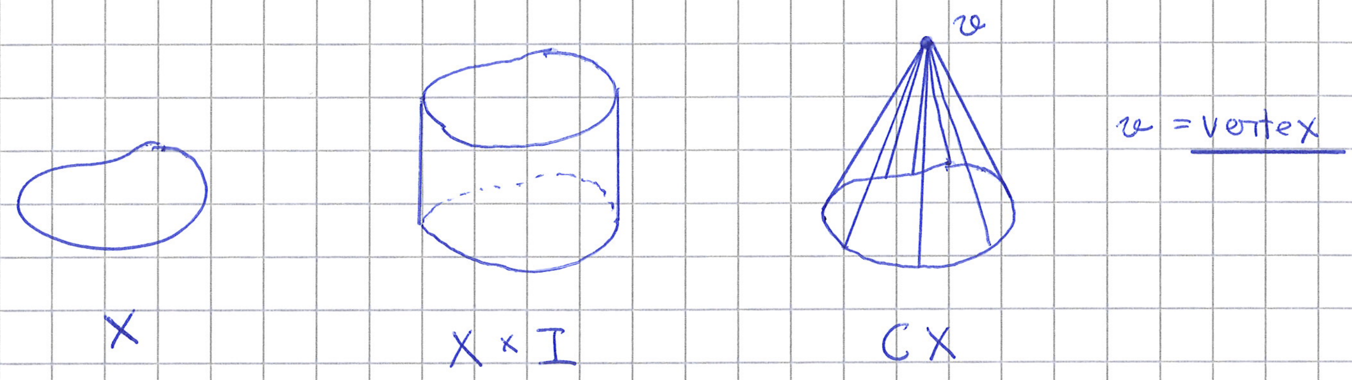
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "\nu"'] & & Z \\ & X/\ker h \arrow[ur, "\varphi"'] & \end{tikzcd}$$

By an earlier example, φ is an injection. Since h is a surjection, also φ is a surjection. Since ν is an identification, it follows from Corollary 3.9 that φ is continuous. To show that φ is a homeomorphism, it suffices to show that it is an open map. Let U be an open subset of $X/\ker h$. Then

$$h^{-1}(\varphi(U)) = (\varphi \circ \nu)^{-1}(\varphi(U)) = \nu^{-1}(\varphi^{-1}(\varphi(U))) = \nu^{-1}(U)$$

is open in X, since ν is continuous. Since h is an identification, it follows that $\varphi(U)$ is open in Z. Thus φ is a homeomorphism. $\square$

Definition 3.11. Let X be a topological space. The relation $(x,t) \sim (x',t')$ if $t = t' = 1$ is an equivalence relation on $X \times I$. The quotient space $X \times I/\sim$ is called the <u>cone over X</u> and denoted by CX.



Example 3.12. Let X and Y be topological spaces and let $y_0 \in Y$. Let $f: X \times I \to Y$ satisfy $f(x,1) = y_0$ for all $x \in X$. Then f induces a continuous map

$$\bar{f}: CX \to Y, \quad [x,t] \mapsto f(x,t).$$

Choose $X = S^n$, $Y = D^{n+1}$ and

$$f: S^n \times I \to D^{n+1}, \quad (u,t) \mapsto (1-t)u.$$

Then $f(u,1) = 0$ for all $u \in S^n$ and there is a continuous map

$$\bar{f}: CS^n \to D^{n+1}, \quad [u,t] \mapsto (1-t)u.$$

Homework: Check that $\bar{f}$ is a homeomorphism.

Thus we may consider D^{n+1} as the cone over S^n with vertex 0.

Theorem 3.13. The cone CX is contractible for every topological space X.

proof. Let $F: CX \times I \to CX$, $([x,t],s) \mapsto [x,(1-s)t+s]$.

Then F is continuous, $F_0 = 1_{CX}$ and F_1 equals the constant map taking every point of CX to the vertex point. □

Theorem 3.14. A topological space X is contractible if and only if it has the same homotopy type as a point.

proof. Let $\{a\}$ be a one-point space. Assume first that X and $\{a\}$ have the same homotopy type. Then there are maps $f: X \to \{a\}$ and $g: \{a\} \to X$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_{\{a\}}$ (in fact, $f \circ g = 1_{\{a\}}$). Now, g is the map $\{a\} \to X$, $a \mapsto x_0$, for some $x_0 \in X$. Thus $(g \circ f)(x) = g(f(x)) = g(a) = x_0$ for all $x \in X$. Hence $g \circ f$ is a constant map and $1_X \simeq g \circ f$ is nullhomotopic.

Assume then that X is contractible. Then $1_X \simeq k$ where $k: X \to X$, $x \mapsto x_0$, for some $x_0 \in X$. Let $f: X \to \{x_0\}$ be the constant map and let $g: \{x_0\} \to X$, $x_0 \mapsto x_0$. Then $f \circ g = 1_{\{x_0\}}$ and $g \circ f = k \simeq 1_X$. Thus X and $\{x_0\}$ have the same homotopy type. □

4. Paths and Path Connectedness

Definition 4.1. A path in a topological space is a continuous map $f: I \to X$. If $f(0) = a$ and $f(1) = b$, we say that f is a path from a to b.

Notice that if f is a path in X from a to b, then $g(t) = f(1-t)$ is a path in X from b to a.

Definition 4.2. A topological space X is path connected if, for every $a, b \in X$, there is a path from a to b.

Theorem 4.3. If X is path connected, then X is connected.

proof. See Rotman Theorem 1.14 or almost any text book in general topology. □

Example 4.4. The subset $\{(x, \sin\frac{1}{x}) \mid 0 < x \leq \frac{1}{2\pi}\} \cup \{(0,y) \mid -1 \leq y \leq 1\}$ of $\mathbb{R}^2$ is connected but not path connected. This subset is called the topologist's sine curve.

Theorem 4.5. Let X be a topological space. Define a relation $\sim$ on X by setting $a \sim b$ if there is a path in X from a to b. Then $\sim$ is an equivalence relation.

proof. Rotman, Theorem 1.15 □

Definition 4.6. The equivalence classes of X under the relation $\sim$ in Theorem 4.5 are called the path components of X.

Definition 4.7. Let $\pi_0(X)$ denote the set of path components of X. If $f: X \to Y$, define $\pi_0(f): \pi_0(X) \to \pi_0(Y)$ to be the function taking a path component C of X to the unique path component of Y containing $f(C)$.

Theorem 4.8. $\pi_0 : Top \to Sets$ is a functor. If $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.

proof. Clearly, π_0 preserves the identity and composition. Therefore, π_0 is a functor.

Let $f, g : X \to Y$ and assume $F : f \simeq g$. Let C be a path component of X. Then $C \times I$ is path connected. Since F is continuous, also $F(C \times I)$ is path connected. Now

$$f(C) = F(C \times \{0\}) \subset F(C \times I)$$

and

$$g(C) = F(C \times \{1\}) \subset F(C \times I).$$

Thus the unique path component of Y containing $F(C \times I)$ contains both $f(C)$ and $g(C)$. Hence $\pi_0(f) = \pi_0(g)$. $\square$

Corollary 4.9. If X and Y have the same homotopy type, then they have the same number of path components.

proof. Let $f : X \to Y$ and $g : Y \to X$ and assume that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Then

$$\pi_0(g) \circ \pi_0(f) = \pi_0(g \circ f) = \pi_0(1_X) = 1_{\pi_0(X)}, \text{ and}$$

$$\pi_0(f) \circ \pi_0(g) = \pi_0(f \circ g) = \pi_0(1_Y) = 1_{\pi_0(Y)}.$$

Thus $\pi_0(f)$ is a bijection. $\square$

<u>Definition 4.10.</u> A topological space X is called <u>locally path connected</u>, if the following holds: For every $x \in X$ and for every open neighborhood U of x there is an open subset V of X with $x \in V \subset U$ such that any two points in V can be joined by a path in U!

<u>Example 4.11.</u> Let X be the set

$$\{(x, \sin \tfrac{1}{x}) \mid 0 < x \leq \tfrac{1}{2\pi}\} \cup \{(0,y) \mid -1 \leq y \leq 1\} \cup A,$$

where A is the line segment joining the points $(0,1)$ and $(\frac{1}{2\pi}, 0)$. Then X is path connected but not locally path connected.

<u>Theorem 4.12.</u> A topological space X is locally path connected if and only if the path components of open subsets are open. In particular, if X is locally path connected, then its path components are open.

<u>proof.</u> Assume first that X is locally path connected. Let U be an open subset of X and let C be a path component of U. Let $x \in C$. Then there is an open subset V of X such that $x \in V \subset U$ and every point in V can be joined to x by a path in U. Thus every point in V is in the same path component as x. Hence $V \subset C$. Therefore, C is open.

Assume then that the path components of open subsets of X are open. Let U be an open subset of X, let $x \in U$ and let V be the path component of x in U. Then V is open and it follows that X is locally path connected. □

Corollary 4.13. A topological space X is locally path connected if and only if, for every $x \in X$ and for every open neighborhood U of x, there is an open path connected V with $x \in V \subset U$.

proof. If X is locally path connected, then one can choose V to be the path component of U containing x. The converse is clear. □

Corollary 4.14. Let X be a locally path connected topological space. Then the components of every open set coincide with its path components. In particular, the components of X coincide with the path components of X.

proof. Let U be an open subset of X and let C be a component of U. Let $\{A_j \mid j \in J\}$ be the path components of C. Then C is the disjoint union of the A_j. By Theorem 4.12, the A_j are open in X. Therefore, they are open in C. Since the complement of any A_j in C is the union of the , $i \neq j$, it follows that the A_j are closed in C. Since C is connected, it follows that it must have exactly one path component. □

Corollary 4.15. If X is connected and locally path connected, then X is path connected.

Definition 4.16. Let X be a topological space and let $A \subset X$. Let $i : A \hookrightarrow X$ be the inclusion. If there is a continuous $r : X \to A$ with $r \circ i = 1_A$ and $i \circ r \simeq 1_X$, we call A a <u>deformation retract</u> of X.

Thus A is a deformation retract of X if there is a continuous $F: X \times I \to X$ with the following properties:

1) $F(x,0) = x$ for all $x \in X$,
2) $F(x,1) \in A$ for all $x \in X$,
3) $F(a,1) = a$ for all $a \in A$.

Theorem 4.17. If A is a deformation retract of X, then A and X have the same homotopy type. $\square$

Corollary 4.18. The circle S^1 is a deformation retract of $\mathbb{C} - \{0\}$, the spaces S^1 and $\mathbb{C} - \{0\}$ have the same homotopy type.

proof. Write each $z \in \mathbb{C} - \{0\}$ in polar coordinates:

$$z = \rho e^{i\theta}, \quad \rho > 0, \quad 0 \leq \theta < 2\pi.$$

Let $F: (\mathbb{C} - \{0\}) \times I \to \mathbb{C} - \{0\}$, $(\rho e^{i\theta}, t) \mapsto [(1-t)\rho + t] e^{i\theta}$.

Then F satisfies conditions 1–3 above. Thus S^1 is a deformation retract of $\mathbb{C} - \{0\}$. $\square$

For topological spaces A and B, let $A \amalg B$ denote their disjoint union topologized so that both A and B are open subsets of $A \amalg B$.

Definition 4.19. Let $f: X \to Y$ be continuous. The mapping cylinder of f is

$$M_f = ((X \times I) \amalg Y) / \sim,$$

where $(x,t) \sim y$ if $y = f(x)$ and $t = 1$.