Let then X be any topological space, not necessarily convex. Let $\sigma: \Delta^n \to X$ be an n-simplex. Define

$$T_n(\sigma) = \sigma_\# \underbrace{T_n(\delta^n)}_{\in S_{n+1}(\Delta^n)} \in S_{n+1}(X),$$

and extend T_n by linearity,

$$T_n\left(\sum_i n_i \sigma_i\right) = \sum_i n_i T_n(\sigma_i), \quad \text{for } \sigma_i : \Delta^n \to X.$$

Then

$$\partial_{n+1} T_n \sigma - T_{n-1} \partial_n \sigma$$

$$\overset{\text{def}}{=} \partial_{n+1} \sigma_\# T_n(\delta^n) - T_{n-1} \partial_n \sigma_\#(\delta^n)$$

$$= \sigma_\# \partial_{n+1} T_n(\delta^n) - T_{n-1} \partial_n \sigma_\#(\delta^n) \quad (\sigma_\# \text{ is a chain map})$$

$$= \sigma_\# \left(\delta^n - S\partial_n(\delta^n) + T_{n-1} \partial_n(\delta^n) \right) - T_{n-1} \partial_n \sigma_\#(\delta^n)$$

$$= \sigma - \underbrace{\sigma_\# S\partial_n(\delta^n)}_{= S\partial_n(\sigma)} + \sigma_\# T_{n-1} \partial_n(\delta^n) - T_{n-1} \sigma_\# \partial_n(\delta^n)$$

(the claim holds for Δn)

$$= \sigma - S\partial_n(\sigma), \quad \text{since } \underbrace{\sigma_\# T_{n-1} = T_{n-1} \sigma_\#}_{\substack{\text{This is true by the} \\ \text{following commutative} \\ \text{diagram}}}$$

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_\#} & S_n(Y) \\ {\scriptstyle T_n}\downarrow & & \downarrow{\scriptstyle T_n} \\ S_{n+1}(X) & \xrightarrow[f_\#]{} & S_{n+1}(Y) \end{array}$$

$f: X \to Y$

$\sigma: \Delta n \to X$

$$T_n f_\#(\sigma) = T_n(f \circ \sigma) = (f \circ \sigma)_\# T_n(\delta^n)$$

$$f_\# T_n(\sigma) = f_\# \circ \sigma_\# T_n(\delta^n) = (f \circ \sigma)_\# T_n(\delta^n).$$

$\square$

Corollary 21.6. Let $q \geq 0$ and let $z \in Z_n(X)$. Then $[z] = [Sd^q z]$.

proof. Since $H_n(Sd): H_n(X) \to H_n(X)$ is the identity, it follows that $[Sd\,z] = [z]$. Then $[Sd^2 z] = [Sd\,z] = [z]$ and, inductively, $[Sd^q z] = [z]$. □

Definition 21.7. Let E be a subspace of a euclidean space and let $\gamma = \sum m_i \sigma_i \in S_n(E)$ (finite sum, $m_i \neq 0\ \forall i$), where $\sigma_i : \Delta^n \to E\ \forall i$. We say that

1) σ_i is <u>affine</u>, if $\sigma_i(\sum t_j e_j) = \sum t_j \sigma_i(e_j)$, where $t_j \geq 0\ \forall j$ and $\sum t_j = 1$, and
2) γ is <u>affine</u>, if σ_i is affine $\forall i$.

Recall Theorem 5.16.: Let S be the n-simplex $[p_0, \dots, p_n]$, and let b be the barycenter of S. Then

$$\| b - p_i \| \leq \frac{n}{n+1} \operatorname{diam}(S),$$

where $\operatorname{diam} S = \sup\{\|u - v\| \mid u, v \in S\}$ (definition)

$$= \sup_{i,j} \{\|p_i - p_j\|\} \quad (\text{Thm } 5.16, \text{ part } 2)$$

Let $\gamma = \sum m_i \sigma_i$ as in Definition 21.7. Since Δ_n is compact and σ_i is continuous, the image $\sigma_i(\Delta_n)$ is a compact subset of a euclidean space. We define $\operatorname{diam}(\sigma_i)$ to mean the diameter of the image $\sigma_i(\Delta^n)$.

This proof is a mess and should be rewritten. Sorry!

Lemma 21.8. Let E be a subspace of a euclidean space. Let $q \geq 1$ and let $\chi : \Delta_q \to E$ be an affine singular q-simplex. Let $Sd(\chi) = \sum m_i \sigma_i \in S_q(E)$. Then

$$\operatorname{diam}(\sigma_i) \leq \frac{q}{q+1} \operatorname{diam}(\chi) \quad \forall i.$$

proof. 1) Prove the result in the case where χ is the identity simplex $\delta^q : \Delta^q \to \Delta^q$.

2) Write $Sd(\delta^q) = \sum n_j \tau_j$, $\tau_j : \Delta^q \to \Delta^q$.
$\Rightarrow Sd(\chi) = \chi_\# Sd(\delta^q) = \sum n_j \chi \circ \tau_j$.

3) Show that, for every j,

$$\operatorname{diam}(\tau_j) \leq \frac{q}{q+1} \operatorname{diam}(\delta^q)$$

implies that

$$\operatorname{diam}(\chi \circ \tau_j) \leq \frac{q}{q+1} \operatorname{diam}(\underbrace{\chi \circ \delta^q}_{=\chi}).$$

(1) Follows immediately from 5.16. $\delta^q(\Delta^q) = \Delta^q$.
If $Sd\, \delta^q = \sum m_i \sigma_i$, then

$\sigma_i(\Delta^q)$

$$\operatorname{diam}(\sigma_i) = \operatorname{diam}(\sigma_i(\Delta^q)) \overset{5.16}{=} \sup_{k,j} \{ \|x_k - x_j\| \mid x_k, x_j \text{ vertices of } \sigma_i(\Delta^q) \}$$
$$= \sup_k \{ \|x_k - b^q\| \mid x_k \text{ vertex of } \sigma_i(\Delta^q) \}$$
$$\leq \frac{q}{q+1} \operatorname{diam}(\Delta^q) \,)$$

Details of the proof are left as an exercise. $\square$

Proposition 21.9. Let E be a subspace of a euclidean space. Let $q \geq 1$ and let $\chi : \Delta_q \to E$ be an affine singular q-simplex. Then, for all $p \geq 1$,

$$\operatorname{diam}(\sigma_i) \leq \left(\frac{q}{q+1}\right)^p \operatorname{diam}(\chi),$$

where σ_i is any singular simplex in the linear combination of $Sd^p(\chi)$.

proof. Apply Lemma 21.8 repeatedly. $\square$

Lemma 21.10. Let X be a topological space and let X_1 and X_2 be subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$. Let $\gamma : \Delta_n \to X$. Then there is $p \geq 1$, with

$$Sd^p(\gamma) \in S_n(X_1) + S_n(X_2).$$

proof. Since $\gamma : \Delta^n \to X$ is continuous, it follows that the sets $\gamma^{-1}(\mathring{X}_1)$ and $\gamma^{-1}(\mathring{X}_2)$ form an open cover for Δ^n. Now, Δ^n is a compact metric space. By the Lebesgue number theorem (Theorem 10.1), there exists $\lambda > 0$ such that whenever $x, y \in \Delta^n$, $\|x-y\| < \lambda$, then

$$\{x,y\} \subset \gamma^{-1}(\mathring{X}_1) \quad \text{or} \quad \{x,y\} \subset \gamma^{-1}(\mathring{X}_2).$$

Let $p \geq 1$ be such that

$$\left(\frac{n}{n+1}\right)^p \operatorname{diam} \Delta^n < \lambda.$$

The identity $\delta^n : \Delta^n \to \Delta^n$ is an affine n-simplex. By Proposition 21.9,

$$\operatorname{diam}(\sigma_j) \leq \left(\frac{n}{n+1}\right)^p \underbrace{\operatorname{diam}(\delta^n)}_{=\operatorname{diam}(\Delta^n)} < \lambda,$$

for any singular simplex σ_j in the linear combination of $Sd^p(\delta^n)$. Write $Sd^p(\delta^n) = \sum m_j \sigma_j$. Then $\operatorname{diam}(\sigma_j(\Delta_n)) < \lambda$, for every j. Thus, for every j,

$$\sigma_j(\Delta_n) \subset \gamma^{-1}(\mathring{X}_1) \quad \text{or} \quad \sigma_j(\Delta_n) \subset \gamma^{-1}(\mathring{X}_2).$$

Now,

$$Sd^p \gamma = \gamma^\# Sd^p(\delta^n) = \gamma^\#\left(\sum m_j \sigma_j\right) = \sum m_j \gamma\sigma_j,$$

Thus

$$\gamma\sigma_j(\Delta_n) \subset \mathring{X}_1 \subset X_1 \quad \text{or} \quad \gamma\sigma_j(\Delta_n) \subset \mathring{X}_2 = X_2,$$

for every j. Thus we can write $Sd^p\gamma = \gamma_1 + \gamma_2$, where $\gamma_1 \in S_n(X_1)$ and $\gamma_2 \in S_n(X_2)$ (collect terms). $\square$

Lemma 21.11. Let X be a topological space and let X_1 and X_2 be subspaces of X. Assume the inclusion

$$S_*(X_1)+S_*(X_2) \to S_*(X)$$

induces isomorphisms in homology. Then excision holds for the subspaces X_1 and X_2 of X. (This means that the inclusion

$$j:(X_1, X_1\cap X_2) \to (X_1\cup X_2, X_2) = (X, X_2)$$

induces isomorphisms

$$j_*: H_n(X_1, X_1\cap X_2) \to H_n(X, X_2), \quad \text{for all } n\geq 0.)$$

proof. There is a short exact sequence of chain complexes:

$$0 \to \underbrace{S_*(X_1)+S_*(X_2)}_{K} \to \underbrace{S_*(X)}_{L} \to \underbrace{S_*(X)/(S_*(X_1)+S_*(X_2))}_{M} \to 0,$$

and it induces a long exact sequence in homology:

$$\cdots \to H_n(K) \xrightarrow{f_n} H_n(L) \to H_n(M) \to H_{n-1}(K) \xrightarrow{f_{n-1}} H_{n-1}(L) \to \cdots$$

By assumption, the sequence has isomorphisms f_n, for all n. It follows that

$$H_*(S_*(X)/(S_*(X_1)+S_*(X_2))) = H_*(M) = 0, \text{ for all } n.$$

There also is a short exact sequence of chain complexes:

$$0 \to \underbrace{\frac{S_*(X_1)+S_*(X_2)}{S_*(X_2)}}_{K'} \xrightarrow{k} \underbrace{\frac{S_*(X)}{S_*(X_2)}}_{L'} \to \underbrace{\frac{S_*(X)}{S_*(X_1)+S_*(X_2)}}_{M} \to 0,$$

and it induces a long exact sequence in homology:

$$\cdots \to H_n(K') \xrightarrow{H_n(k)} H_n(L') \to \underset{\underset{0}{\wr\!\!\parallel}}{H_n(M)} \to H_{n-1}(K') \to H_{n-1}(L') \to \cdots$$

Since $H_n(M) = 0 \ \forall n$, it follows that

$$k_* : H_*(K') \to H_*(L')$$

is an isomorphism, for all $*$.

Consider the commutative diagram

$$\begin{tikzcd}
\dfrac{S_*(X_1)}{S_*(X_1 \cap X_2)} \arrow[rr, "j"] \arrow[dr, "l"'] & & \dfrac{S_*(X)}{S_*(X_2)} \\
& \dfrac{S_*(X_1) + S_*(X_2)}{S_*(X_2)} \arrow[ur, "k"'] &
\end{tikzcd}$$

where j is induced by the inclusion

$$(X_1, X_1 \cap X_2) \to (X, X_2)$$

Also l is an isomorphism (this you can see by considering the elements in $S_*(X)/S_*(X_1 \cap X_2)$ and in $(S_*(X_1) + S_*(X_2))/S_*(X_2)$, both elements can be identified with $\sum m_i \sigma_i$, where $\sigma_i : \Delta^* \to X_1 - X_2$, or by using the isomorphism theorem: Let S'_* and S''_* be subcomplexes of S_*. Then there is an isomorphism

$$S'_* / (S'_* \cap S''_*) \to (S'_* + S''_*)/S''_*.)$$

Since $kl = j$, it follows that

$$k_* l_* = j_* : H_*(\quad) \to H_*(\quad)$$

Since l_* and k_* are isomorphisms, it follows that

$$j_* : H_*(X_1, X_1 \cap X_2) \to H_*(X, X_2)$$

is an isomorphism. Therefore, excision holds for X_1 and X_2 in X. $\square$

Theorem 21.12 (the proof for Excision 2)

Let X be a topological space and let X_1 and X_2 be subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$. Then the inclusion

$$j : (X_1, X_1 \cap X_2) \to (X, X_2)$$

induces isomorphisms

$$j_* : H_n(X_1, X_1 \cap X_2) \to H_n(X, X_2), \text{ for all } n \geq 0.$$

proof. By Lemma 21.11, it is sufficient to show that the inclusion

$$S_*(X_1) + S_*(X_2) \hookrightarrow S_*(X)$$

induces isomorphisms

$$\Theta_n : H_n(S_*(X_1) + S_*(X_2)) \to H_n(S_*(X)) = H_n(X),$$

for all n. Let $\gamma_1 \in S_n(X_1)$, $\gamma_2 \in S_n(X_2)$ and assume $\gamma_1 + \gamma_2$ is a cycle in $S_n(X_1) + S_n(X_2)$. Then $\gamma_1 + \gamma_2$ is also a cycle in $S_n(X)$. Denote the homology class of $\gamma_1 + \gamma_2$ in $S_n(X_1) + S_n(X_2)$ by $[\gamma_1 + \gamma_2]_{sub}$ and in $S_n(X)$ by $[\gamma_1 + \gamma_2]$, as usual. Then

$$\Theta_n : [\gamma_1 + \gamma_2]_{sub} \mapsto [\gamma_1 + \gamma_2].$$

Θ_n is surjective: Let $z = \sum m_i z_i \in Z_n(X) \subset S_n(X)$, where $z_i : \Delta_n \to X$ are continuous for all (finitely many) i. Then, by Lemma 21.10, there is $q \geq 1$ such that

$$Sd^q(z_i) = \gamma_i^1 + \gamma_i^2,$$

where $\gamma_i^1 \in S_n(X_1)$ and $\gamma_i^2 \in S_n(X_2)$, for every i. Since z is an n-cycle and since Sd^q is a chain map, it follows that

$$Sd^q z = Sd^q(\sum m_i z_i) = \sum m_i(\gamma_i^1 + \gamma_i^2)$$

is an n-cycle. Therefore,

$$[\sum m_i(\gamma_i^1 + \gamma_i^2)]_{sub} \in H_n(S_*(X_1) + S_*(X_2))$$

and

$$\Theta_n([\sum m_i(\gamma_i^1+\gamma_i^2)]_{sub}) = [\underbrace{\overbrace{\sum m_i(\gamma_i^1+\gamma_i^2)}^{Sd^q(z_i)}}_{Sd^q(z)}]$$
$$= [Sd^q(z)] = [z].$$

$\therefore$ Θ_n is surjective

Θ_n is injective: Assume $\Theta[\gamma_1+\gamma_2]_{sub} = 0 = [\gamma_1+\gamma_2]$. Then $\exists \beta \in S_{n+1}(X) : \partial\beta = \gamma_1 + \gamma_2$. It follows from Lemma 21.10 that there is $q \geq 1$: $Sd^q\beta = \beta_1 + \beta_2$, where $\beta_1 \in S_{n+1}(X_1)$ and $\beta_2 \in S_{n+1}(X_2)$. Then

$$\partial(\beta_1+\beta_2) = \partial Sd^q\beta = Sd^q\partial\beta = Sd^q(\gamma_1+\gamma_2).$$

↑ Sd^q is a chain map

Thus $[Sd^q(\gamma_1+\gamma_2)]_{sub} = 0$.

Let $j_i : X_i \hookrightarrow X$, $i=1,2$, be the inclusion.

By Lemma 21.3, the diagram

$$\begin{tikzcd}
S_n(X_i) \arrow[r, "Sd_n"] \arrow[d, "(j_i)_\#"'] & S_n(X_i) \arrow[d, "(j_i)_\#"] \\
S_n(X) \arrow[r, "Sd_n"'] & S_n(X)
\end{tikzcd}$$

commutes for $i=1,2$. Thus $Sd: S_*(X) \to S_*(X)$ takes $S_*(X_1)$ into $S_*(X_1)$ and $S_*(X_2)$ into $S_*(X_2)$. Therefore, Sd takes the subcomplex $S_*(X_1) + S_*(X_2)$ into $S_*(X_1) + S_*(X_2)$.

Similarly, the diagram

$$\begin{tikzcd}
S_n(X_i) \arrow[r, "T_n"] \arrow[d, "(j_i)_\#"'] & S_{n+1}(X_i) \arrow[d, "(j_i)_\#"] \\
S_n(X) \arrow[r, "T_n"'] & S_{n+1}(X)
\end{tikzcd}$$

commutes for $i=1,2$. Thus the chain homotopy $\{T_n: S_n(X) \to S_{n+1}(X)\}$ restricts to chain homotopies $\{T_n^1 : S_n(X_1) \to S_{n+1}(X_1)\}$ and $\{T_n^2 : S_n(X_2) \to S_{n+1}(X_2)\}$.

Hence

$$\gamma_1 - Sd^q \gamma_1 = (T^1\partial + \partial T^1)\,\gamma_1$$

and

$$\gamma_2 - Sd^q \gamma_2 = (T^2\partial + \partial T^2)\,\gamma_2 .$$

Thus

$$\gamma_1 + \gamma_2 - Sd^q(\gamma_1 + \gamma_2) = \underbrace{T^1\partial\gamma_1 + T^2\partial\gamma_2}_{T\partial(\gamma_1+\gamma_2)} + \partial(T^1\gamma_1 + T^2\gamma_2)$$

since T^1, T^2 are restrictions of T.

Now, $\partial(T^1\gamma_1 + T^2\gamma_2) \in B_n(S_*(X_1) + S_*(X_2))$.

Since $[\gamma_1 + \gamma_2] = 0$, it follows that $T\underbrace{\partial(\gamma_1+\gamma_2)}_{=0} = 0$.

Thus $[\gamma_1+\gamma_2]_{\text{sub}} = [Sd^q(\gamma_1+\gamma_2)]_{\text{sub}}$

$= [\partial(\beta_1+\beta_2)]_{\text{sub}} = 0.$

$\therefore\ \Theta$ is an injection. $\square$

22. Some applications

Let X_α, $\alpha \in I$, be topological spaces. Let $x_\alpha \in X_\alpha$ be a basepoint of X_α, for all α. The wedge sum (or just the wedge) of the X_α is

$$\bigvee_\alpha X_\alpha = \coprod_\alpha X_\alpha/\sim, \qquad (\coprod = \text{disjoint union})$$

where $\sim$ is the equivalence relation on $\coprod_\alpha X_\alpha$ obtained by identifying all the basepoints x_α with each others. Write

$$X = \coprod_\alpha X_\alpha \qquad \text{and} \qquad A = \coprod_\alpha \{x_\alpha\}.$$

Then

$$X/A = \bigvee_\alpha X_\alpha.$$

Proposition 22.1. Let X_α, $\alpha \in I$, be pathconnected, topological spaces with basepoints x_α, respectively. Assume each x_α is a strong deformation retract of an open subset U_α of X_α. Then, for all n,

$$\tilde{H}_n(\bigvee_\alpha X_\alpha) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha).$$

proof. For every $\alpha \in I$, let $F_\alpha : U_\alpha \times I \to U_\alpha$ be a strong deformation retraction to x_α. Then

$$F_\alpha(x,0) = x, \quad \text{for all} \quad x \in U_\alpha$$
$$F_\alpha(x,1) = x_\alpha, \quad \text{for all} \quad x \in U_\alpha$$
$$F_\alpha(x_\alpha,t) = x_\alpha, \quad \text{for all} \quad t \in I.$$

Let

$$F : \bigcup_\alpha U_\alpha \times I \to \bigcup_\alpha U_\alpha,$$

$$F(x,t) = F_\alpha(x,t) \quad \forall (x,t) \in U_\alpha \times I.$$

Then
$$F(x,0) = x, \quad \text{for all} \quad x \in \bigcup_\alpha U_\alpha$$
$$F(x,1) \in A = \bigcup_\alpha \{x_\alpha\}, \quad \text{for all} \quad x \in \bigcup_\alpha U_\alpha$$
$$F(x_\alpha,t) = x_\alpha, \quad \text{for all } x_\alpha \text{ and for all } t.$$

Thus F strongly deformation retracts $\bigcup_\alpha U_\alpha$ to A. It follows that, for all $n \geq 0$,

$$\tilde{H}_n(\bigvee_\alpha X_\alpha) = \tilde{H}_n(\bigcup_\alpha X_\alpha / A)$$

$$\cong H_n(\bigcup_\alpha X_\alpha, A) \quad (\text{Theorem } 20.9)$$

$$\cong \bigoplus_\alpha H_n(X_\alpha, \underbrace{A \cap X_\alpha}_{=\{x_\alpha\}}) \quad (*)$$

$$= \bigoplus_\alpha H_n(X_\alpha, x_\alpha)$$

$$\cong \bigoplus_\alpha \tilde{H}_n(X_\alpha) \quad (\text{Theorem } 20.1)$$

□

(*) follows from the following:

<u>Theorem 22.2.</u> Let X be a topological space with path components X_α, $\alpha \in I$. Let $A \subset X$. For every $n \geq 0$,

$$H_n(X,A) \cong \bigoplus_\alpha H_n(X_\alpha, A \cap X_\alpha).$$

<u>proof</u>: See Rotman, Theorem 5.13. □

Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a homomorphism. Then, for some $m \in \mathbb{Z}$, $\phi(n) = mn$, for all n, i.e., ϕ is multiplication by some integer m.

Definition 22.3. Let $n > 0$ and let $\phi : S^n \to S^n$ be continuous. The map ϕ has degree $m \in \mathbb{Z}$, if $\phi_\# : H_n(S^n) \to H_n(S^n)$ is multiplication by m.

Notation: If ϕ has degree m, we write $d(\phi) = m$.

Earlier (Definition 9.5) we defined degree $\deg(\phi)$ for closed paths $\phi : I \to S^1$, i.e., for maps $\phi : S^1 \to S^1$. One can show that for $\phi : S^1 \to S^1$, $\deg(\phi) = d(\phi)$.
(The proof uses the Hurewicz map $\pi_1(S^1, 1) \to H_1(S^1)$
(See Rotman, Theorem 6.20.)

The following lemma lists some basic properties of the degree:

Lemma 22.4. Let $\phi, g : S^n \to S^n$ be continuous, $n > 0$. Then

1) $d(g \circ \phi) = d(g) d(\phi)$,
2) $d(id_{S^n}) = 1$,
3) If ϕ is constant, then $d(\phi) = 0$,
4) If $\phi \simeq g$, then $d(\phi) = d(g)$
5) If ϕ is a homotopy equivalence, then $d(\phi) = \pm 1$.

Proof.

1) Let $d(g) = p$, $d(\phi) = m$. Then

$$\mathbb{Z} \xrightarrow{\phi_\#} \mathbb{Z} \xrightarrow{g_\#} \mathbb{Z} \qquad \phi_\#(n) = mn \qquad g_\#(\phi_\#(n)) = p(mn) = (pm)n$$

$$g_\#(\phi_\#(n)) = (g \circ \phi)_\#(n)$$

$$\Rightarrow d(g \circ \phi) = d(g) d(\phi)$$

2) $(id)_\# = id : H_n(S^n) \to H_n(S^n) \Rightarrow d(id) = 1$

3) f constant $\Rightarrow f_\# : H_n(S^n) \to H_n(S^n)$ is the zero map $\Rightarrow d(f) = 0$

4) If $f \simeq g$, then $f_\# = g_\# \Rightarrow d(f) = d(g)$

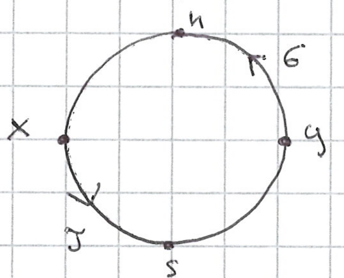
5) Let f be a homotopy equivalence, let g be a homotopy inverse of f. Then

$d(f)d(g) \overset{1}{=} d(f \circ g) \overset{4}{=} d(id) \overset{2}{=} 1$

$d(f), d(g) \in \mathbb{Z} \Rightarrow d(f) = d(g) = 1$ or $d(f) = d(g) = -1$.

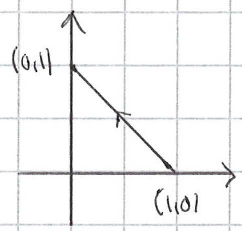
□

Proposition 22.5. Let $x = (-1,0)$ and $y = (1,0)$. Let σ be the (northerly) path in S^1 from y to x, and let τ be the (southerly) path in S^1 from x to y. Then $\sigma + \tau$ is a 1-cycle in S^1 and the homology class $[\sigma + \tau]$ generates $H_1(S^1)$.



$n = (0,1)$ = the north pole
$s = (0,-1)$ = the south pole

Proof. Here, $\sigma, \tau : \Delta^1 \to S^1$.



Then $\partial(\sigma)(1) = \sigma(0,1) - \sigma(1,0) = x - y$
and $\partial(\tau)(1) = \tau(0,1) - \tau(1,0) = y - x$

$\Rightarrow \partial(\sigma+\tau)(1) = \partial(\sigma)(1) + \partial(\tau)(1) = (x-y) + (y-x) = 0$

$\Rightarrow \partial(\sigma+\tau) = 0 \Rightarrow \sigma + \tau$ is a 1-cycle.

To prove that $[\sigma + \tau]$ generates $H_1(S^1)$ we need a couple of lemmas: